

Chap 4

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} \quad (1-\beta^2) = \frac{1}{\gamma^2}$$

$$\begin{aligned} \gamma \frac{d}{dt} [\gamma(c, \underline{v})] &= \gamma^4 \frac{\underline{v} \cdot \underline{a}}{c^2} [c, \underline{v}] + \gamma^2 \frac{d}{dt} (c, \underline{v}) \\ &= \gamma^4 \frac{\underline{v} \cdot \underline{a}}{c^2} (c, \underline{v}) + \gamma^2 (0, \underline{a}) \\ &= \gamma^2 \left(\gamma^2 \frac{\underline{v} \cdot \underline{a}}{c^2}, \underline{a} + \gamma^2 (\underline{v} \cdot \underline{a}) \frac{\underline{v}}{c^2} \right) \end{aligned}$$

$$A_\alpha A^\alpha = -\gamma^4 \left(\gamma^4 \frac{(\underline{v} \cdot \underline{a})^2}{c^2} \right) + \gamma^4 \left(a^2 + 2(\underline{a} \cdot \underline{v}) \frac{\gamma^2 (\underline{v} \cdot \underline{a})}{c^2} + \gamma^4 \frac{(\underline{a} \cdot \underline{v})^2 v^2}{c^2} \right)$$

$$= \gamma^4 \left\{ -\gamma^4 \frac{(\underline{a} \cdot \underline{v})^2}{c^2} + a^2 + 2\gamma^2 \frac{(\underline{a} \cdot \underline{v})^2}{c^2} + \gamma^4 \frac{(\underline{a} \cdot \underline{v})^2 v^2}{c^4} \right\}$$

$$= \gamma^4 \left\{ a^2 + \frac{(\underline{a} \cdot \underline{v})^2}{c^2} \gamma^2 \left[\gamma^2 \beta^2 - \gamma^2 + 2 \right] \right\}$$

$$= \gamma^4 \left\{ a^2 + \frac{\gamma^2 (\underline{a} \cdot \underline{v})^2}{c^2} \left[\gamma^2 (\beta^2 - 1) + 2 \right] \right\}$$

$$\frac{-\gamma^2 (1-\beta^2) + 2}{-1 + 2}$$

$$= \gamma^4 \left\{ a^2 + \frac{\gamma^2 (\underline{a} \cdot \underline{v})^2}{c^2} \right\}$$

Section 40 Material Stress - Energy Tensor

$$(\rho v^i)_{,t} + \pi^i_{,j} = f^i$$

$$V^\alpha = (-c, \underline{v})$$

$$V^\beta = (-c, \underline{v})$$

$$(V^\alpha V^\beta)_0 = (+c^2, 0)$$

$$\begin{aligned} (P^{\alpha\beta})_0 &= \eta^{\alpha\beta} + c^{-2} (V^\alpha V^\beta)_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

$$M_0 = \begin{pmatrix} \rho_{00} c^2 & & & \\ & p & & 0 \\ & 0 & p & \\ & & & p \end{pmatrix}$$

$$\rho_{00} = \rho_0 (1 + v^2/c^2) = \rho_0 + \rho_0 v^2/c^2$$

$$\rho_{00} c^2 = \rho_0 c^2 + \rho_0 v^2$$

$$M_0 = \cancel{\rho_{00}} \rho_0 (V^\alpha V^\beta)_0 + p \left[\eta^{\alpha\beta} + c^{-2} (V^\alpha V^\beta)_0 \right]$$

$$M_0 = (V^\alpha V^\beta)_0 (\rho_{00} + p/c^2) + \eta^{\alpha\beta} p$$

$$V^\alpha = \gamma(c, \underline{v}) \quad V_\alpha = \gamma(-c, \underline{v})$$

$$V^i = \gamma v^i \quad V^i V^j = \gamma^2 v^i v^j$$

$$\rho_{000} = \rho_{00} + p/c^2 = \rho_0 (1 + e/c^2) + p/c^2$$

$$= \rho_0 (1 + e/c^2 + (p/\rho_0)/c^2)$$

$$\rho_{000} = \rho_0 (1 + h/c^2)$$

$$M^{\alpha\beta} = \rho_{000} V^\alpha V^\beta + p \eta^{\alpha\beta}$$

$$= \rho_0 (1 + e/c^2 + p/\rho_0 c^2) V^\alpha V^\beta + p \eta^{\alpha\beta}$$

$$M^{ij} = \rho_0 (1 + e/c^2 + p/\rho_0 c^2) \gamma^2 v^i v^j + p \delta^{ij}$$

$$\gamma^2 = \frac{1}{(1-\beta^2)}$$

$$\gamma^2 - 1 = \frac{1}{(1-\beta^2)} - 1 = \frac{1 - (1-\beta^2)}{1-\beta^2} = \frac{\beta^2}{1-\beta^2}$$

$$\gamma = (1 - v^2/c^2)^{-1/2}$$

$$= 1 + \frac{1}{2} \frac{v^2}{c^2} + \dots$$

$$\gamma (\gamma \rho_0) v^i v^j (1 + e/c^2 + p/\rho_0 c^2) + p \delta^{ij}$$

$$= \frac{\beta^2}{1-\beta^2} \gamma \rho v^i v^j + \gamma \rho (v^i v^j / c^2) (e + p/\rho_0) + p \delta^{ij}$$

$$M^{0j} = (\rho_{00} + p/c^2) (V^0 \gamma v^j) + p \delta^{0j}$$

$$= \rho_0 (1 + e/c^2 + p/\rho_0 c^2) \gamma^2 c v^j$$

$$= \gamma c \rho v^j (1 + e/c^2 + p/\rho_0 c^2)$$

$$M^{00} = (\rho_{00} + p/c^2) \gamma^2 c^2 + p(-1)$$

$$= \gamma^2 c^2 \rho_0 (1 + e/c^2) + \gamma^2 c^2 p/c^2 - p = \gamma^2 c^2 \rho_0 (1 + e/c^2) + \gamma^2 c^2 p/c^2 - p$$

$$M^{\infty} = \gamma^2 c^2 \rho_0 (1 + e/c^2) + \beta^2 \gamma^2 p$$

$$= \gamma^2 \rho_0 (c^2 + e + p/\rho_0 \beta^2)$$

$$= \gamma^2 \rho_0 (c^2 + e + \beta^2 p/\rho_0)$$

$$= (1 + \frac{1}{2} v^2/c^2) \rho_0 (c^2 + e + (p/\rho_0) v^2/c^2)$$

$$= \rho c^2 + \rho e + \frac{1}{2} \rho v^2 +$$

Section 42

$$F^\mu \equiv \Phi^\mu / \delta v_0 = (\gamma / \delta v_0) (\underline{\phi} \cdot \underline{v} / c, \underline{\phi})$$

$$F^\mu = (\underline{f} \cdot \underline{v} / c, \underline{f})$$

$$M^{\alpha\beta}_{;\beta} = F^\alpha$$

$$(\rho_{000} v^\alpha v^\beta + p g^{\alpha\beta})_{;\beta} = F^\alpha$$

Cartesian $(\rho_{000} v^\alpha v^\beta + p \eta^{\alpha\beta})_{;\beta} = F^\alpha$

~~M^{\alpha\beta}~~ $M^{ij} = \rho_0 \gamma^2 v^i v^j (1 + e/c^2 + p/\rho_0 c^2) + p \delta^{ij}$

$$M^{0i} = M^{i0} = \gamma^2 \rho_0 c v^i (1 + e/c^2 + p/\rho_0 c^2)$$

$$M^{00} = \gamma^2 \rho_0 (c^2 + e + \beta^2 p/\rho_0)$$

or $M^{ij} = \rho_{000} \gamma^2 v^i v^j + p \delta^{ij} = \rho_i v^i v^j + p \delta^{ij}$

~~$M^{ij} = \rho_{000} \gamma^2 v^i v^j + p \delta^{ij} = (\rho_i v^i v^j) + (p \delta^{ij})$~~

$$M^{i0} = M^{0i} = \gamma^2 c v^i \rho_{000} = \rho_i c v^i$$

$$M^{00} = \rho_{000} (\gamma^2 c^2) - p = c^2 \rho_i - p$$

$$(M^{0\beta})_{;\beta} = (M^{00})_{;t} + (M^{0i})_{;i} = \frac{1}{c} (M^{00})_{;t} + (M^{0i})_{;i}$$

$$= \frac{1}{c} (c^2 \rho_i - p)_{;t} + (\rho_i c v^i)_{;i} = \underline{f} \cdot \underline{v} / c$$

$$= (c \rho_i - p/c)_{;t} + (\rho_i v^i)_{;i} = c [(\rho_i - p/c)_{;t} + (\rho_i v^i)_{;i}] = \underline{f} \cdot \underline{v} / c$$

$$\begin{aligned}
 (m^{i\beta})_{j\beta} &= (m^{i0})_{,t} + (m^{ij})_{,j} \\
 &= \frac{1}{c} (\rho_1 c v^i)_{,t} + (\rho_1 v^i v^j + p \delta^{ij})_{,j} \\
 (42.4a) \quad &= (\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p \delta^{ij})_{,j} = f^i
 \end{aligned}$$

$$\begin{aligned}
 p_1 &= \gamma^2 p_{000} = \gamma^2 p_0 (1 + e/c^2 + p/\rho_0 c^2) \\
 &= \gamma^2 (p_0 + \rho_0 e/c^2 + p/c^2)
 \end{aligned}$$

$$c^2 [(\rho_1 - p/c^2)_{,t} + (\rho_1 v^j)_{,j}] = \underline{f} \cdot \underline{v}$$

$$(42.5) \Rightarrow [\gamma^2 (\rho_0 c^2 + \rho_0 e + p) - p]_{,t} + [\gamma^2 (\rho_0 c^2 + \rho_0 e + p) v^j]_{,j} = \underline{f} \cdot \underline{v}$$

$$(\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p \delta^{ij})_{,j} = f^i$$

$$[\gamma^2 (\rho_0 c^2 + \rho_0 e + p) v^i]_{,t} + [\gamma^2 (\rho_0 c^2 + \rho_0 e + p) v^i v^j + c^2 p \delta^{ij}]_{,j} = f^i c^2$$

$$(42.6) \quad \quad \quad + [\gamma^2 (\rho_0 c^2 + \rho_0 e + p) v^i v^j]_{,j} + c^2 p_{,j} = f^i c^2$$

$$\eta^{\alpha\beta} V_\alpha = (-V^0, \underline{V})$$

Gas Energy Eqn

$$(42.2) \quad (\rho_{000} V^\alpha V^\beta + p \eta^{\alpha\beta})_{;\beta} = F^\alpha$$

$$V_\alpha (\rho_{000} V^\alpha V^\beta + p \eta^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha$$

$$= V_\alpha V^\alpha (\rho_{000} V^\beta)_{;\beta} + V_\alpha \rho_{000} V^\beta V^\alpha_{;\beta} + V_\alpha (p g^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha$$

$$= -c^2 (\rho_{000} V^\beta)_{;\beta} + \rho_{000} V^\beta V_\alpha V^\alpha_{;\beta} + \eta^{\alpha\beta} V_\alpha p_{;\beta} = V_\alpha F^\alpha$$

$$(42.8) \quad = -c^2 (\rho_{000} V^\alpha)_{;\alpha} + V^\alpha p_{;\alpha} = V_\alpha F^\alpha$$

$$(\rho_0 V_\alpha)_{;\alpha} (c^2 + e + P/\rho_0) = 0$$

$$-c^2 (\rho_{000} V^\alpha)_{;\alpha} + (\rho_0 V_\alpha)_{;\alpha} (c^2 + e + P/\rho_0) + V^\alpha p_{;\alpha} = V_\alpha F^\alpha$$

$$= - [\rho_0 (c^2 + e + P/\rho_0) V^\alpha]_{;\alpha} + (c^2 + e + P/\rho_0) (\rho_0 V_\alpha)_{;\alpha} + V^\alpha p_{;\alpha} = V_\alpha F^\alpha$$

$$= - (c^2 + e + P/\rho_0) (\rho_0 V^\alpha)_{;\alpha} - \rho_0 V^\alpha (c^2 + e + P/\rho_0)_{;\alpha} + (c^2 + e + P/\rho_0) (\rho_0 V_\alpha)_{;\alpha} + V^\alpha p_{;\alpha} = V_\alpha F^\alpha$$

$$- \rho_0 V^\alpha (c^2 + e + P/\rho_0)_{;\alpha} + V^\alpha p_{;\alpha} = V_\alpha F^\alpha$$

$$- \rho_0 V^\alpha e_{;\alpha} - \rho_0 V^\alpha \frac{\partial}{\partial x^\alpha} \left(\frac{P}{\rho_0} \right) + V^\alpha \frac{\partial}{\partial x^\alpha} (P) = V_\alpha F^\alpha$$

$$- \rho_0 V^\alpha \frac{\partial e}{\partial x^\alpha} - \cancel{V^\alpha \frac{\partial P}{\partial x^\alpha}} - \frac{\rho_0 V^\alpha P}{\rho_0^2} \frac{\partial \rho_0}{\partial x^\alpha} + \cancel{V^\alpha \frac{\partial P}{\partial x^\alpha}} = V_\alpha F^\alpha$$

$$(42.10) \quad - \rho_0 V^\alpha \frac{\partial e}{\partial x^\alpha} + \frac{P}{\rho_0} V^\alpha \frac{\partial \rho_0}{\partial x^\alpha} = V_\alpha F^\alpha = \rho_0 V^\alpha \left[- \frac{\partial e}{\partial x^\alpha} + \frac{P}{\rho_0^2} \frac{\partial \rho_0}{\partial x^\alpha} \right]$$

$$- \left\{ \begin{aligned} &(\rho, v_i)_{,t} + (\rho, v_i v^j)_{,j} + p_{,i} = f_i \\ &v_i \left((\rho - \rho/c^2)_{,t} + v_i (\rho, v^j)_{,j} \right) = \left(\underline{f} \cdot \underline{v} / c^2 \right) v_i \end{aligned} \right\}$$

$$\begin{aligned} &(\rho, v_i)_{,t} - v_i (\rho)_{,t} + v_i (p/c^2)_{,t} + p_{,i} \\ &+ (\rho, v_i v^j)_{,j} - v_i (\rho, v^j)_{,j} = f_i - v_i \underline{f} \cdot \underline{v} / c^2 \end{aligned}$$

$$p_{,i} + \rho, v_i, t + v_i (p/c^2)_{,t} + \rho, v^j, v_{i,j} = f_i - v_i \underline{f} \cdot \underline{v} / c^2$$

$$(42.12) \quad \rho, (v_i, t + v^j, v_{i,j}) + \frac{v_i}{c^2} p_{,t} + p_{,i} = f_i - v_i \underline{f} \cdot \underline{v} / c^2$$

$$\text{Now } v^\alpha \frac{\partial}{\partial x^\alpha} = \frac{D}{Dt} = \gamma c \frac{\partial}{\partial t} + \gamma v_i \frac{\partial}{\partial x^i} = \gamma \frac{\partial}{\partial t} + \gamma v_i \frac{\partial}{\partial x^i}$$

$$\text{and } \underline{p}_i \left(\frac{\partial v_i}{\partial t} + v^j \frac{\partial v_i}{\partial x^j} \right) = \underline{p}_i \left(\frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j} \right) v_i = \frac{\rho_i}{\gamma} \frac{D v_i}{Dt} = \rho_i^* \frac{D v_i}{Dt}$$

$$\text{hence } (42.13) \quad \rho_i^* \frac{D v_i}{Dt} + \frac{v_i}{c^2} \nabla_i p = f_i - v_i \underline{f} \cdot \underline{v} / c^2$$

$$\rho_i^* \frac{D v_i}{Dt} = -\nabla p + \underline{f} - \underline{v} (\underline{f} \cdot \underline{v}) / c^2 - \underline{v} p_{,t} / c^2$$

$$(42.13) \quad = -\nabla p + \underline{f} - (\underline{v} / c^2) [\underline{f} \cdot \underline{v} + \partial p / \partial t]$$

$\int p_0 f(p_0) dp_0 = ?$ for isotropic $f(p_0)$

$$\int E f(p) dp = \quad dp = \frac{E}{E_0} dp_0$$

$$\int (E^2/E_0) f(p_0) dp_0$$

$$= \gamma \int \left[\frac{E_0 + \underline{v} \cdot \underline{p}_0 / E_0}{E_0} \right]^2 f(p_0) dp_0$$

$$= \gamma^2 \int E_0 [1 + \underline{v} \cdot \underline{p}_0 / E_0]^2 f(p_0) dp_0$$

$$= \gamma^2 \int \left[\frac{E_0 + \underline{v} \cdot \underline{p}_0}{E_0} \right]^2 f(p_0) dp_0$$

$$= \gamma^2 \int [E_0^2 + 2E_0(\underline{v} \cdot \underline{p}_0) + (\underline{v} \cdot \underline{p}_0)^2] E_0^{-1} f(p_0) dp_0$$

$$= \gamma^2 \int \left\{ E_0 + (\underline{v} \cdot \underline{p}_0)^2 / E_0 \right\} f(p_0) dp_0$$

43.29

$$p = \gamma m_0 v \quad E = \gamma m_0 c^2$$

$$p = E v / c^2$$

$$\text{or } p_0^i = E_0 U_0^i / c^2$$

$$p_0 = E_0 \underline{U}_0 / c^2$$

$$\underline{p} = \underline{p}_0 + \underline{v} \left[\gamma (E_0/c^2) + (\gamma-1) \underline{v} \cdot \underline{p}_0 / v^2 \right]$$

$$\begin{pmatrix} E/c \\ \underline{p} \end{pmatrix} = \begin{pmatrix} \gamma & + \gamma \beta \\ + \gamma \beta & \underline{I} + (\gamma-1) \frac{\beta^2 \beta \beta}{v^2} \end{pmatrix} \begin{pmatrix} E_0/c \\ \underline{p}_0 \end{pmatrix}$$

$$\underline{p} = (+\gamma \beta) E_0/c + \left[\underline{I} + (\gamma-1) \frac{\beta^2 \beta \beta}{v^2} \right] \underline{p}_0$$

$$= \underline{p}_0 + \cancel{\frac{v}{c} E_0}$$

$$+ \underline{v} \gamma E_0/c^2 + (\gamma-1) \frac{v}{v^2} \frac{(\underline{v} \cdot \underline{p}_0) \underline{v}}{c^2}$$

$$\begin{matrix} (43.35) \\ (43.11) \end{matrix} \quad \underline{p} = \underline{p}_0 + \underline{v} \left[\gamma E_0/c^2 + (\gamma-1) (\underline{v} \cdot \underline{p}_0) / v^2 \right] \quad ($$

$$(43.3A) MD = \int \underline{p} f(\underline{p}) d\underline{p} = \int f(\underline{p}_0) \left[1 + (\underline{v} \cdot \underline{p}_0) / E_0 \right]$$

$$\cdot \left\{ \underline{p}_0 + \underline{v} \left[\gamma E_0/c^2 + (\gamma-1) (\underline{v} \cdot \underline{p}_0) / v^2 \right] \right\} d\underline{p}_0$$

=

$$p_{00} c^2 = \int E_0 f_0 d\underline{p}_0$$

$$p_m = \int (\underline{v}_0 \cdot \underline{e}_m) (\underline{p}_0 \cdot \underline{e}_m) f_0 d\underline{p}_0$$

$$MD = \frac{\gamma}{c^2} \int \left[\gamma E_0 \underline{v} + (\gamma-1) v^2 (\underline{v}_0 \cdot \underline{v}) (\underline{p}_0 \cdot \underline{v}) \underline{v} + (\underline{v}_0 \cdot \underline{v}) \underline{p}_0 \right] f_0 d\underline{p}_0$$

$$= \frac{\gamma}{c^2} \left\{ \gamma \underline{v} \int E_0 f_0 d\underline{p}_0 + (\gamma-1) \underline{v} \int (\underline{v}_0 \cdot \underline{e}_m) (\underline{p}_0 \cdot \underline{e}_m) + \underline{v} \int (\underline{v}_0 \cdot \underline{e}_m) \underline{p}_0 f_0 d\underline{p}_0 \right.$$

$$\left. = \underline{v} \left[\gamma v p_{00} / c^2 + (\gamma-1) v D_m + \frac{1}{2} v D_m \right] \right\}$$

$$\boxed{\frac{\partial p}{\partial z} = \rho g}$$

$$\int (\frac{\partial u}{\partial z}) \rho_0 f_0 d\rho_0$$

$$= \rho \int (\frac{\partial u}{\partial z}) \rho_0 f_0 d\rho_0$$

$$(43.39) \quad \frac{\gamma}{c^2} \{ \gamma v_{\perp} \rho_{00} c^2 + \gamma v_{\perp} p_m \} = \gamma^2 \{ v_{\perp} \rho_{00} + v_{\perp} p_m / c^2 \}$$

$$= \gamma^2 \rho_{000} v_{\perp}$$

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial z} = \rho g$$

$$\frac{Du}{Dt} = - \frac{\partial p}{\partial m} + g$$

$$(43.45) \quad \Pi^{ij} = \int v^i p^j f(\rho) d\rho = \int v^i p^j f_0(\rho_0) \frac{E}{E_0} d\rho_0$$

$$\frac{E}{E_0} = \gamma [1 + \frac{v \cdot p_0}{E_0}]$$

$$= \int \frac{p^j}{E_0} (v^i E) f_0(\rho_0) d\rho_0 = c^2 \int \frac{p^i p^j}{E_0} f_0 d\rho_0$$

$$= c^2 \int \frac{f_0 d\rho_0}{E_0} \left[p_0^i + \frac{(\gamma-1)}{v^2} (v \cdot p_0) v^i + \left(\frac{\gamma E_0}{c^2} \right) v^i \right] \times$$

$$\left[p_0^j + \frac{(\gamma-1)}{v^2} (v \cdot p_0) v^j + \left(\frac{\gamma E_0}{c^2} \right) v^j \right]$$

$$= c^2 \int \frac{f_0 d\rho_0}{E_0} \left\{ p_0^i p_0^j + \frac{(\gamma-1)}{v^2} (v \cdot p_0) [p_0^j v^i + p_0^i v^j] + \left(\frac{\gamma E_0}{c^2} \right) [p_0^i v^j + p_0^j v^i] \right.$$

$$\left. + \frac{(\gamma-1)^2}{v^4} (v \cdot p_0)^2 v^i v^j + \left(\frac{\gamma E_0}{c^2} \right) \left(\frac{\gamma-1}{v^2} (v \cdot p_0) \right) [v^i v^j + v^j v^i] \right\}$$

$$= c^2 \int \frac{f_0 d\rho_0}{E_0} \left\{ p_0^i p_0^j + \left[\frac{(\gamma-1)}{v^2} (v \cdot p_0) + \frac{\gamma E_0}{c^2} \right] [p_0^j v^i + p_0^i v^j] \right.$$

$$\left. + v^i v^j \left[\frac{(\gamma-1)^2}{v^4} (v \cdot p_0)^2 + 2 \frac{\gamma E_0}{c^2} \frac{(\gamma-1)}{v^2} (v \cdot p_0) \right] + \frac{\gamma^2 E_0^2}{c^4} v^i v^j \right\}$$

$$\Pi^{ij} = \int \frac{c^2}{E_0} (f_0 d\rho_0) \left\{ \rho_0^i \rho_0^j + \frac{(\gamma-1)(\underline{v} \cdot \underline{\rho}_0)}{v^2} (\rho_0^j v^i + \rho_0^i v^j) + \frac{(\gamma-1)^2}{v^4} (\underline{v} \cdot \underline{\rho}_0)^2 v^i v^j + \frac{\gamma^2 E_0^2}{c^4} v^i v^j \right\}$$

$$c^2 \rho_0 = E_0 \underline{U}_0 \quad \text{hence} \quad \frac{c^2}{E_0} \rho_0 = \underline{U}_0$$

and

$$\begin{aligned} \Pi^{ij} &= \int \underline{U}_0^i \rho_0^j f_0 d\rho_0 + \int \frac{(\gamma-1)(\underline{v} \cdot \underline{U}_0)}{v^2} (\rho_0^j v^i + \rho_0^i v^j) f_0 d\rho_0 \\ &+ \int \frac{(\gamma-1)^2 (\underline{v} \cdot \underline{\rho}_0)(\underline{v} \cdot \underline{U}_0)}{v^4} v^i v^j f_0 d\rho_0 + \int \frac{\gamma^2 E_0}{c^2} v^i v^j f_0 d\rho_0 \end{aligned}$$

$$\begin{aligned} &= \left(\int \underline{U}_0^i \rho_0^j f_0 d\rho_0 + \frac{(\gamma-1)v^j}{v} \int (\underline{U}_0 \cdot \underline{e}) \rho_0^i f_0 d\rho_0 + (\gamma-1) \frac{v^i}{v} \int (\underline{U}_0 \cdot \underline{e}) \rho_0^j f_0 d\rho_0 \right. \\ &+ \left. v^i v^j \frac{(\gamma-1)^2}{v^2} \int (\underline{U}_0 \cdot \underline{e})(\underline{\rho}_0 \cdot \underline{e}) f_0 d\rho_0 + \frac{\gamma^2 v^i v^j}{c^2} \int E_0 f_0 d\rho_0 \right) \\ &= -T^{ij} + \gamma^2 \rho_{00} v^i v^j + (\gamma-1)^2 \frac{p_m v^i v^j}{v^2} \\ &+ (\gamma-1) \frac{v^j}{v} \int (\underline{U}_0 \cdot \underline{e}) \rho_0^i f_0 d\rho_0 + (\gamma-1) \frac{v^i}{v} \int (\underline{U}_0 \cdot \underline{e}) \rho_0^j f_0 d\rho_0 \end{aligned}$$

(43.46)

$$= + p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + (\gamma-1)^2 \frac{p_m v^i v^j}{v^2}$$

$$+ (\gamma-1) \frac{v^j}{v} \int (\underline{U}_0 \cdot \underline{e})(\underline{\rho}_0 \cdot \underline{e}) \frac{v^i}{v} f_0 d\rho_0 + (\gamma-1) \frac{v^i}{v} \int (\underline{U}_0 \cdot \underline{e})(\underline{\rho}_0 \cdot \underline{e}) \frac{v^j}{v} f_0 d\rho_0$$

$$= p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + [(\gamma-1)^2 + 2(\gamma-1)] p_m \frac{v^i v^j}{v^2}$$

$$\begin{aligned} (\gamma-1)^2 + 2(\gamma-1) &= \gamma^2 - 2\gamma + 1 + 2\gamma - 2 \\ &= \gamma^2 - 1 = \frac{1}{1-\beta^2} - 1 = \frac{1 - (1-\beta^2)}{1-\beta^2} = \frac{\beta^2 \gamma^2}{1-\beta^2} = \gamma^2 \frac{v^2}{c^2} \end{aligned}$$

(43.47)

$$\Pi^{ij} = p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + \gamma^2 \frac{v^2}{c^2} p_m \frac{v^i v^j}{v^2} = p_m \delta^{ij} + \gamma^2 v^i v^j \left[\rho_{00} + \frac{p_m}{c^2} \right]$$

Finally!!

$$(43.47) \quad \pi^{ij} = p_m \delta^{ij} + \gamma^2 v^i v^j p_{000}$$

$$p_{00} c^2 = \int E_0 f_0 d^3 p_0 = m_0 \int (c^2 + \frac{1}{2} u_0^2) f_0 d^3 p_0$$

$$= m_0 c^2 n_0 + \frac{m_0}{2} \int u_0^2 f_0 d^3 p_0$$

but for $f_0 = \text{Maxwellian}$, $\langle u_0^2 \rangle = 3kT$

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}$$

$$E_0 = \gamma m_0 c^2$$

$$p_0 = \gamma m_0 v$$

$$E_0^2 = p_0^2 c^2 + m_0^2 c^4$$

p.41

$$g_{ij} g^{jk} = \delta_i^k$$

HL/ITU

$$\begin{aligned} P_{\beta}^{\alpha} P_{\gamma}^{\beta} &= (\delta_{\beta}^{\alpha} + c^{-2} V^{\alpha} V_{\beta}) (\delta_{\gamma}^{\beta} + c^{-2} V^{\beta} V_{\gamma}) \\ &= \delta_{\beta}^{\alpha} \delta_{\gamma}^{\beta} + c^{-2} [\delta_{\beta}^{\alpha} V^{\beta} V_{\gamma} + \delta_{\gamma}^{\beta} V^{\alpha} V_{\beta}] + \underbrace{c^{-4} V^{\alpha} V_{\beta} V^{\beta} V_{\gamma}} \\ &= \delta_{\gamma}^{\alpha} + c^{-2} [V^{\alpha} V_{\gamma} + V^{\alpha} V_{\gamma}] - c^{-2} V^{\alpha} V_{\gamma} \quad V_{\beta} V^{\beta} = -c^2 \end{aligned}$$

$$P_{\beta}^{\alpha} P_{\gamma}^{\beta} = \delta_{\gamma}^{\alpha} + c^{-2} V^{\alpha} V_{\gamma} = P_{\gamma}^{\alpha}$$

$$\begin{aligned} P^{\alpha\beta} &= P^{\alpha\gamma} P_{\gamma}^{\beta} = (g^{\alpha\gamma} + c^{-2} V^{\alpha} V^{\gamma}) (\delta_{\gamma}^{\beta} + c^{-2} V^{\beta} V_{\gamma}) \\ &= g^{\alpha\gamma} \delta_{\gamma}^{\beta} + c^{-2} (\delta_{\gamma}^{\beta} V^{\alpha} V^{\gamma} + g^{\alpha\gamma} V^{\beta} V_{\gamma}) \\ &\quad + c^{-4} V^{\alpha} V^{\beta} V^{\gamma} V_{\gamma} \end{aligned}$$

$$= g^{\alpha\beta} + c^{-2} [V^{\alpha} V^{\beta} + V^{\beta} V^{\alpha}] - c^{-2} V^{\alpha} V^{\beta}$$

$$= g^{\alpha\beta} + c^{-2} V^{\alpha} V^{\beta}$$

$$P^{\alpha\beta} P_{\alpha\beta} = (g^{\alpha\beta} + c^{-2} V^{\alpha} V^{\beta}) (g_{\alpha\beta} + c^{-2} V_{\alpha} V_{\beta})$$

$$= (g^{\alpha\beta} g_{\alpha\beta}) + c^{-2} (g_{\alpha\beta} V^{\alpha} V^{\beta} + g^{\alpha\beta} V_{\alpha} V_{\beta}) + c^{-4} V^{\alpha} V^{\beta} V_{\alpha} V_{\beta}$$

$$= (g^{\alpha\beta} g_{\alpha\beta}) + c^{-2} (V^{\beta} V_{\beta} + \cancel{g^{\alpha\beta} V^{\beta} V_{\beta}}) + c^{-4} (V^{\alpha} V_{\alpha} V^{\beta} V_{\beta})$$

$$P^{\alpha\beta} P_{\alpha\beta} = \delta_{\alpha}^{\alpha} + c^{-2}(-c^2 - c^2) + c^{-4}(c^4)$$

$$= 4 - 2 + 1 = 3$$

$$V_{\alpha} P_{\beta}^{\alpha} = V_{\alpha} [\delta_{\beta}^{\alpha} + c^{-2} V^{\alpha} V_{\beta}] = V_{\beta} + c^{-2}(-c^2 V_{\beta}) = 0$$

$$A^{\alpha} = (-c^2 V_{\alpha} A^{\beta} V^{\alpha}) + P_{\beta}^{\alpha} A^{\beta}$$

$$= (-c^2 V^{\alpha} V_{\alpha} A^{\beta}) + P_{\beta}^{\alpha} A^{\beta}$$

$$A^{\alpha} = (-c^2 V_{\beta} A^{\beta}) V^{\alpha} + P_{\beta}^{\alpha} A^{\beta}$$

$$= (-c^2 V^{\alpha} V_{\beta} A^{\beta}) + [\delta_{\beta}^{\alpha} + c^{-2} V^{\alpha} V_{\beta}] A^{\beta}$$

$$= \delta_{\beta}^{\alpha} A^{\beta} + c^{-2} V^{\alpha} V_{\beta} A^{\beta} - c^{-2} V^{\alpha} V_{\beta} A^{\beta}$$

$$= \delta_{\beta}^{\alpha} A^{\beta} = A^{\alpha}$$

$$W^{\alpha\beta} = w^{\alpha} V^{\beta} + w^{\beta} V^{\alpha} + w^{\alpha\beta}$$

$$= (c^{-4} V_{\gamma} V_{\gamma} w^{\delta\gamma}) V^{\alpha} V^{\beta} + (-c^2 P_{\delta}^{\alpha} w^{\delta\gamma} V_{\gamma}) V^{\beta} + (-c^2 P_{\delta}^{\beta} w^{\delta\gamma} V_{\gamma}) V^{\alpha}$$

$$+ P_{\delta}^{\alpha} P_{\delta}^{\beta} w^{\delta\gamma}$$

$$P_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} + c^{-2} V^{\alpha} V_{\beta}$$

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$$P_{\beta}^{\gamma} = \delta_{\beta}^{\gamma} + c^{-2} V^{\gamma} V_{\beta}$$

$$\begin{aligned} & V_{\alpha;\gamma} [\delta_{\beta}^{\gamma} + c^{-2} V^{\gamma} V_{\beta}] = V_{\alpha;\beta} + c^{-2} V^{\gamma} V_{\beta} V_{\alpha;\gamma} \\ & = (c^{-4} V_{\delta} V_{\gamma} W^{\delta\gamma}) V^{\alpha} V^{\beta} - c^{-2} (\delta_{\delta}^{\alpha} + c^{-2} V^{\alpha} V_{\delta}) W^{\delta\gamma} V_{\gamma} V^{\beta} \\ & \quad - c^{-2} (\delta_{\delta}^{\beta} + c^{-2} V^{\beta} V_{\delta}) W^{\delta\gamma} V_{\gamma} V^{\alpha} + (\delta_{\delta}^{\alpha} + c^{-2} V^{\alpha} V_{\delta}) (\delta_{\delta}^{\beta} + c^{-2} V^{\beta} V_{\delta}) W^{\delta\gamma} \\ & = c^{-4} (\cancel{V^{\alpha} V_{\delta}}) (\cancel{V^{\beta} V_{\gamma}}) W^{\delta\gamma} - c^{-2} (W^{\alpha\gamma} V_{\delta} V^{\beta} + c^{-2} V^{\alpha} V_{\delta} \cancel{V^{\beta} V_{\gamma}} W^{\delta\gamma}) \\ & \quad - c^{-2} (W^{\beta\gamma} V_{\delta} V^{\alpha} + c^{-2} W^{\delta\gamma} \cancel{V^{\beta} V_{\delta}} V^{\alpha} V_{\gamma}) + \delta_{\delta}^{\alpha} \delta_{\delta}^{\beta} W^{\delta\gamma} + c^{-2} V^{\alpha} V_{\delta} \delta_{\delta}^{\beta} W^{\delta\gamma} \\ & \quad + \delta_{\delta}^{\alpha} c^{-2} V^{\beta} V_{\delta} W^{\delta\gamma} + c^{-4} V^{\alpha} V_{\delta} \cancel{V^{\beta} V_{\gamma}} W^{\delta\gamma} \\ & = -c^{-2} \cancel{V_{\delta} V^{\beta} W^{\alpha\gamma}} - c^{-2} \cancel{V_{\delta} V^{\alpha} W^{\beta\gamma}} + W^{\alpha\beta} + c^{-2} \cancel{V^{\alpha} V_{\delta} W^{\beta\gamma}} \\ & \quad + c^{-2} \cancel{V^{\beta} V_{\delta} W^{\alpha\gamma}} \\ & = W^{\alpha\beta} \end{aligned}$$

$$V_{\alpha;\beta} V^{\beta} = (sV_{\alpha}/s\tau) = A_{\alpha}$$

$$\begin{aligned} V_{\alpha;\gamma} P_{\beta}^{\gamma} &= V_{\alpha;\gamma} [\delta_{\beta}^{\gamma} + c^{-2} V^{\gamma} V_{\beta}] \\ &= V_{\alpha;\beta} + c^{-2} V_{\beta} V^{\gamma} V_{\alpha;\gamma} = V_{\alpha;\beta} + c^{-2} V_{\beta} A_{\alpha} \end{aligned}$$

$$\Rightarrow V_{\alpha;\beta} = V_{\alpha;\gamma} P_{\beta}^{\gamma} - c^{-2} V_{\beta} A_{\alpha}$$

$$\omega_{\alpha\beta} = \frac{1}{2} [V_{\alpha;\gamma} P_{\beta}^{\gamma} - V_{\beta;\gamma} P_{\alpha}^{\gamma}]$$

$$e'_{\alpha\beta} = \frac{1}{2} [V_{\alpha;\gamma} P_{\beta}^{\gamma} + V_{\beta;\gamma} P_{\alpha}^{\gamma}]$$

$$V_{\alpha;\gamma} P_{\beta}^{\gamma} = \omega_{\alpha\beta} + e'_{\alpha\beta} = V_{\alpha;\beta} + c^{-2} A_{\alpha} V_{\beta}$$

$$\text{or } V_{\alpha;\beta} = e'_{\alpha\beta} + \omega_{\alpha\beta} - c^{-2} A_{\alpha} V_{\beta}$$

Then if $e_{\alpha\beta} = e'_{\alpha\beta} - \frac{1}{3} \Theta P_{\alpha\beta}$

we have

$$V_{\alpha;\beta} = e_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3} \Theta P_{\alpha\beta} - c^{-2} A_{\alpha} V_{\beta}$$

$$V^{\alpha} e_{\alpha\beta} = \frac{V^{\alpha}}{2} (V_{\alpha;\gamma} P_{\beta}^{\gamma} + V_{\beta;\gamma} P_{\alpha}^{\gamma}) - \frac{V^{\alpha}}{3} V_{\gamma;\beta} P_{\alpha}^{\gamma}$$

$$= \frac{1}{2} \{ V^{\alpha} V_{\alpha;\gamma} P_{\beta}^{\gamma} + V^{\alpha} V_{\beta;\gamma} P_{\alpha}^{\gamma} \} - \frac{1}{3} V^{\alpha} V_{\gamma;\beta} P_{\alpha}^{\gamma}$$

$$= \frac{1}{2} \{ V^{\alpha} V_{\alpha;\gamma} (\delta_{\beta}^{\gamma} + c^{-2} V^{\gamma} V_{\beta}) + V^{\alpha} V_{\beta;\gamma} (\delta_{\alpha}^{\gamma} + c^{-2} V^{\gamma} V_{\alpha}) \}$$

$$- \frac{1}{3} V^{\alpha} V_{\gamma;\beta} [g_{\alpha\beta} + c^{-2} V_{\alpha} V_{\beta}]$$

$$= \frac{1}{2} \{ V^{\alpha} V_{\alpha;\beta} + c^{-2} V^{\alpha} V^{\gamma} V_{\alpha;\gamma} V_{\beta} + V^{\alpha} V_{\beta;\alpha} + c^{-2} V^{\alpha} V_{\alpha}^{\gamma} V^{\delta} V_{\beta;\delta} \}$$

$$- \frac{1}{3} V_{\beta}^{\gamma} V_{\gamma;\alpha} - \frac{1}{3} c^{-2} V^{\alpha} V_{\alpha} V_{\beta} V_{\gamma;\delta}$$

$$= \frac{1}{2} \{ V^{\alpha} V_{\alpha;\beta} + c^{-2} A_{\alpha} V^{\alpha} V_{\beta} + A_{\beta} - V^{\gamma} V_{\beta;\gamma} \} - \frac{1}{3} V_{\beta}^{\gamma} V_{\gamma;\alpha} + \frac{1}{3} V_{\alpha}^{\gamma} V_{\gamma;\beta}$$

$$V_{\alpha} P_{\beta}^{\alpha} = V_{\beta} P_{\alpha}^{\beta} = 0$$

$$V_{\alpha} [S_{\beta}^{\alpha} + c^{-2} V^{\alpha} V_{\beta}] = V_{\beta} + c^{-2} V_{\beta} (V^{\alpha} V_{\alpha}) = 0$$

Now $V^{\alpha} V_{\beta;\alpha} = A_{\beta}$, hence the ~~second~~ ^{2nd + 4th} terms cancel, leaving

$$V^{\alpha} e_{\alpha\beta} = \frac{1}{2} \left\{ V^{\alpha} V_{\alpha;\beta} + c^{-2} A_{\alpha} V^{\alpha} V_{\beta} \right\}$$

$$= \frac{1}{2} \left\{ V^{\beta} V_{\beta;\alpha} + c^{-2} V_{\beta} V^{\alpha} A_{\alpha} \right\} = \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} V^{\alpha} A_{\alpha} \right]$$

$$V_{\alpha} P_{\beta}^{\alpha} = 0$$

$$V_{\alpha} A^{\alpha} = 0$$

$$\text{Now } A_{\alpha} = a V_{\alpha} + a_{\alpha}$$

~~$$V_{\beta;\alpha} P_{\beta}^{\alpha} + c^{-2} A_{\beta} V^{\alpha}$$~~

~~$$V^{\alpha} e_{\alpha\beta} = \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} V^{\alpha} (a V_{\alpha} + g_{\alpha\gamma} P_{\beta}^{\gamma} A^{\beta}) \right]$$~~

~~$$= \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} a V^{\alpha} V_{\alpha} + c^{-2} V^{\alpha} P_{\alpha\beta} A^{\beta} \right]$$~~

~~$$= \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} - a + c^{-2} V^{\alpha} A^{\beta} (g_{\alpha\beta} + c^{-2} V_{\alpha} V_{\beta}) \right]$$~~

~~$$= \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} V_{\alpha} A^{\alpha} + c^{-2} V_{\beta} A^{\beta} + c^{-4} V^{\alpha} V_{\alpha} V_{\beta} A^{\beta} \right]$$~~

~~$$= \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} (V_{\alpha} A^{\alpha} + V_{\beta} A^{\beta} - V_{\beta} A^{\beta}) \right]$$~~

~~$$= \frac{V_{\beta}}{2} \left[V_{\beta;\alpha} + c^{-2} V_{\alpha} A^{\alpha} \right]$$~~

~~find problem~~

$$A^\alpha = V^\beta V_{\alpha;\beta}$$

$$\frac{d}{dt} (V_\alpha V^\alpha) = 2V_\alpha A^\alpha = 0 = V$$

$$\text{Thus } 2V_\alpha (V^\beta V_{\alpha;\beta}) = 2V^\beta V_\alpha V_{\alpha;\beta}$$

~~$$V^\beta V_{\beta;\alpha} = V^\beta [V_{\beta;\gamma} P_\beta^\gamma - c^2 A_\beta V_\alpha] = 2V^\beta$$

$$= -c^2 (V^\beta A_\beta V_\alpha) + V_{\beta;\gamma} (V^\beta P_\beta^\gamma)$$

$$= -c^2 V_\alpha g^{\beta\beta}$$~~

$$V^\alpha V_{\alpha;\beta} = V^\alpha [V_{\alpha;\gamma} P_\beta^\gamma - c^2 A_\alpha V_\beta]$$
~~$$= -c^2 V^\alpha A_\alpha V_\beta + V^\alpha V_{\alpha;\gamma} P_\beta^\gamma$$~~

$$\rightarrow = (V^\alpha V_\alpha)_{;\beta} - V_\alpha V^\alpha_{;\beta} = (V^\alpha V_\alpha)_{;\beta} - g_{\alpha\gamma} V^\gamma (g^{\alpha\delta} V_\delta)_{;\beta}$$

$$= 0 - g_{\alpha\gamma} g^{\alpha\delta} V^\gamma V_{\delta;\beta} + g_{\alpha\gamma} V^\gamma V_\delta (g^{\alpha\delta})_{;\beta}$$

$$V^\alpha V_{\alpha;\beta} = -\sum_\delta V^\delta V_{\delta;\beta} + g_{\alpha\gamma} V^\gamma V_\delta g^{\alpha\delta}_{;\beta}$$

$$= -V^\delta V_{\delta;\beta} + V^\alpha V_\delta g^{\alpha\delta}_{;\beta}$$

$$2V^\alpha V^\beta V_{\alpha;\beta} = 2g_{\alpha\delta} V^\delta g^{\beta\gamma} V_\gamma V_{\alpha;\beta}$$

$$= 2g_{\alpha\delta} g^{\beta\gamma} V_\gamma V^\delta V_{\alpha;\beta}$$

Energy eqn non-ideal fluid

$$M^{\alpha\beta}_{;\beta} = F^\alpha$$

$$M_{\alpha\beta} = \rho_{00} V^\alpha V^\beta + p P_{\alpha\beta} - 2\mu e_{\alpha\beta} + \eta \Theta P_{\alpha\beta} + \bar{c}^2 (Q_\alpha V_\beta + Q_\beta V_\alpha)$$

$$\begin{aligned} (\rho_{00} V^\alpha V^\beta)_{;\beta} &= \cancel{(\rho_{00} V^\alpha V^\beta)_{;\beta}} = V^\beta (\rho_{00})_{;\beta} V^\alpha + \rho_{00} V^\alpha V^\beta_{;\beta} + \rho_{00} V^\beta V^\alpha_{;\beta} \\ &= V^\alpha (\cancel{D\rho_{00}/D\tau}) + \rho_{00} V^\alpha \Theta + \rho_{00} A^\alpha \\ &= V^\alpha [D\rho_{00}/D\tau + \rho_{00} \Theta] + \rho_{00} A^\alpha \end{aligned}$$

~~$$(V^\alpha Q^\beta)_{;\beta} = V^\alpha_{;\beta} Q^\beta + V^\alpha Q^\beta_{;\beta}$$~~

$$(V^\beta Q^\alpha)_{;\beta} = V^\beta Q^\alpha_{;\beta} + V^\beta_{;\beta} Q^\alpha = DQ^\alpha/D\tau + \Theta Q^\alpha$$

$$\begin{aligned} (V^\alpha Q^\beta)_{;\beta} &= V^\alpha_{;\beta} Q^\beta + V^\alpha Q^\beta_{;\beta} \\ &= V^\alpha Q^\beta_{;\beta} + \end{aligned}$$

$$M^{\alpha\beta}_{;\beta} = V^\alpha D\rho_{00}/D\tau + \rho_{00} \Theta V^\alpha + \rho_{00} A^\alpha$$

$$+ \bar{c}^2 \Theta Q^\alpha + (DQ^\alpha/D\tau) \bar{c}^2$$

$$+ Q^\alpha_{;\beta} (e^{\alpha\beta} + \omega^{\alpha\beta}) \bar{c}^2$$

$$+ V^\alpha Q^\beta_{;\beta} \bar{c}^2$$

$$+ p_{;\beta} P^{\alpha\beta} + p \cancel{P^{\alpha\beta}_{;\beta}} + \eta \Theta_{;\beta} P^{\alpha\beta} - \eta \Theta \cancel{P^{\alpha\beta}_{;\beta}} - 2\mu e^{\alpha\beta}_{;\beta}$$

$$+ \frac{1}{3} \Theta Q^\beta_{;\beta} P^{\alpha\beta} \bar{c}^2 - 2\mu e^{\alpha\beta}_{;\beta}$$

$$\begin{aligned}
 (P^{\alpha\beta})_{;\beta} &= (g^{\alpha\beta})_{;\beta} + (\bar{c}^{-2} V^\alpha V^\beta)_{;\beta} = \bar{c}^{-2} V^\alpha_{;\beta} V^\beta + V_\alpha V^\beta_{;\beta} \\
 &= \bar{c}^{-2} (V^\beta V^\alpha_{;\beta} + V_\alpha V^\beta_{;\beta}) \\
 &= \bar{c}^{-2} (A^\alpha + V^\alpha \Theta)
 \end{aligned}$$

$$\begin{aligned}
 m^{\alpha\beta}_{;\beta} &= V^\alpha \left\{ D\rho_{00}/D\tau + \rho_{00} \Theta (\bar{c}^{-2} Q^\beta_{;\beta} + \bar{c}^{-2} \Theta p - \ell \Theta^2 \bar{c}^{-2}) \right\} \\
 &\quad + \rho_{00} A^\alpha + A^\alpha (\bar{c}^{-2} p - \bar{c}^{-2} \ell \Theta) \downarrow - 2\mu e^{\alpha\beta}_{;\beta} + P^{\alpha\beta} (p_{;\beta} - f\theta_{;\beta}) \\
 &\quad + \bar{c}^{-2} \left[\Theta Q^\alpha + Q^\beta_{;\beta} (e^{\alpha\beta} + \omega^{\alpha\beta}) + \frac{1}{3} \Theta Q^\beta_{;\beta} P^{\alpha\beta} + DQ^\alpha/D\tau \right]
 \end{aligned}$$

$$\begin{aligned}
 &= V^\alpha \left\{ D\rho_{00}/D\tau + \Theta [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \right\} + A^\alpha [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \\
 &\quad - 2\mu e^{\alpha\beta}_{;\beta} + \bar{c}^{-2} \left[\Theta Q^\alpha + \frac{1}{3} \Theta Q^\beta_{;\beta} + Q^\beta_{;\beta} (e^{\alpha\beta} + \omega^{\alpha\beta} + \frac{1}{3} \Theta P^{\alpha\beta}) + DQ^\alpha/D\tau \right] \\
 &\quad + P^{\alpha\beta} (p - f\theta)_{;\beta}
 \end{aligned}$$

$$\begin{aligned}
 \Theta Q^\alpha + \frac{1}{3} \Theta Q^\beta_{;\beta} P^{\alpha\beta} &= \Theta Q^\alpha + \frac{1}{3} \Theta Q_\beta [g^{\alpha\beta} + \bar{c}^{-2} V^\alpha V^\beta] \\
 &= \Theta Q^\alpha + \frac{1}{3} \Theta Q_\beta g^{\alpha\beta} + \frac{\bar{c}^{-2}}{3} (V^\beta Q_\beta) V^\alpha \\
 &= \frac{4}{3} \Theta Q^\alpha
 \end{aligned}$$

$$\begin{aligned}
 m^{\alpha\beta}_{;\beta} &= V^\alpha \left\{ D\rho_{00}/D\tau + \Theta [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \right\} + A^\alpha [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \\
 &\quad - 2\mu e^{\alpha\beta}_{;\beta} + \bar{c}^{-2} \left[\frac{4}{3} \Theta Q^\alpha + V^\alpha Q^\beta_{;\beta} + Q_\beta (e^{\alpha\beta} + \omega^{\alpha\beta}) + DQ^\alpha/D\tau \right] \\
 &\quad + P^{\alpha\beta} (p - f\theta)_{;\beta}
 \end{aligned}$$

$$\begin{aligned}
 0 = V_\alpha m^{\alpha\beta}_{;\beta} &= V_\alpha V^\alpha \left\{ D\rho_{00}/D\tau + \Theta [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \right\} + 0 - 2\mu N_\alpha e^{\alpha\beta}_{;\beta} \\
 &\quad + \bar{c}^{-2} [0 + V_\alpha V^\alpha Q^\beta_{;\beta} + 0 + V_\alpha DQ^\alpha/D\tau] + 0
 \end{aligned}$$

$$= -\bar{c}^{-2} \left\{ D\rho_{00}/D\tau + \Theta [\rho_{00} + \bar{c}^{-2} (p - \ell \Theta)] \right\} - 2\mu V_\alpha e^{\alpha\beta}_{;\beta} - Q^\beta_{;\beta} + \bar{c}^{-2} V_\alpha DQ^\alpha/D\tau$$

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$$\begin{aligned}
 V^\alpha \rho_{00;\alpha} &= V^\alpha (\rho_{000} - P/c^2)_{;\alpha} \\
 &= V^\alpha \rho_{000;\alpha} - V^\alpha c^2 P_{;\alpha} + \rho_{000} V^\alpha_{;\alpha} - \rho_{000} V^\alpha_{;\alpha} \\
 &= (\rho_{000} V^\alpha)_{;\alpha} - c^2 V^\alpha P_{;\alpha} - \rho_{000} V^\alpha_{;\alpha}
 \end{aligned}$$

$$c^2 (\rho_{000} V^\alpha)_{;\alpha} - V^\alpha P_{;\alpha} = -2\mu V_\alpha e^{\alpha\beta}_{;\beta} + j\theta^2 = [Q^\beta_{;\beta} + c^2 V_\alpha DQ^\alpha / DT]$$

$$V_\alpha DQ^\alpha / DT = D(V_\alpha Q^\alpha) / D\tau - Q^\alpha DV_\alpha / D\tau = -A_\alpha Q^\alpha$$

$$V_\alpha e^{\alpha\beta}_{;\beta} = (V_\alpha e^{\alpha\beta})_{;\beta} - e^{\alpha\beta} V_{\alpha;\beta} = -e^{\alpha\beta} (e_{\alpha\beta} \omega_{\alpha\beta} + \frac{1}{3} \theta P_\beta - c^2 A_\alpha V_\beta)$$

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$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} + \frac{e^{\alpha\beta} P_\beta}{3} + e^{\alpha\beta} c^2 A_\alpha V_\beta$$

$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} - \frac{e^{\alpha\beta}}{2} (V_{\alpha;\gamma} P_\beta^\gamma - V_{\beta;\gamma} P_\alpha^\gamma)$$

$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} - \frac{1}{2} \{ [(V_\alpha e^{\alpha\beta})_{;\gamma} - V_\alpha e^{\alpha\beta}_{;\gamma}] P_\beta^\gamma - [(V_\beta e^{\alpha\beta})_{;\gamma} - V_\beta e^{\alpha\beta}_{;\gamma}] P_\alpha^\gamma \}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} + \frac{1}{2} \{ V_\alpha e^{\alpha\beta}_{;\gamma} P_\beta^\gamma + V_\beta e^{\alpha\beta}_{;\gamma} P_\alpha^\gamma \}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} + \frac{1}{2} \{ V_\alpha e^{\alpha\beta}_{;\gamma} [\delta_\beta^\gamma + c^2 V^\gamma V_\beta] + V_\beta e^{\alpha\beta}_{;\gamma} [\delta_\alpha^\gamma + c^2 V^\gamma V_\alpha] \}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} \omega_{\alpha\beta} + \frac{1}{2} \{ V_\alpha e^{\alpha\beta}_{;\beta} + c^2 V_\alpha V^\gamma V_\beta e^{\alpha\beta}_{;\beta} + V_\beta e^{\alpha\beta}_{;\alpha} + c^2 e^{\alpha\beta}_{;\gamma} V_\beta V^\gamma V_\alpha \}$$

$$c^2 (\rho_{000} V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = -2\mu V_\alpha e^{\alpha\beta}_{;\beta} + f\theta^2 [Q^\beta_{;\beta} - c^{-2} V_\alpha DQ^\alpha / D\tilde{c}]$$

$$= -2\mu (-e^{\alpha\beta} V_{\alpha;\beta}) + f\theta^2 - (Q^\beta_{;\beta} + c^{-2} A_\alpha Q^\alpha)$$

$$+ 2\mu (+e^{\alpha\beta} e_{\alpha\beta})$$

$$(\rho_{000} c^2 V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = 2\mu e^{\alpha\beta} e_{\alpha\beta} + f\theta^2 - (Q^\beta_{;\beta} + c^{-2} A_\alpha Q^\alpha)$$

$$(\rho_0 S V^\alpha + Q^\alpha / T)_{;\alpha} = \rho_0 \cancel{S} / D\tilde{c} + \cancel{\rho_0 S} (\rho_0 V^\alpha)_{;\alpha}$$

$$+ (Q^\alpha / T)_{;\alpha} \geq 0$$

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$$\frac{1}{c^2} V_\alpha V^\beta Q^\alpha_{;\beta}$$

$$c^{-2} V_\alpha V^\alpha Q^\beta$$

$$-(\rho_{000} c^2 V^\alpha)_{;\alpha} + V^\alpha p_{;\alpha} + f\theta^2 + 2\mu c^{\alpha\beta} e_{\alpha\beta} - [Q^\beta_{;\beta} + c^2 A_\alpha Q^\alpha]$$

$$(\rho_{000} c^2 V^\alpha)_{;\alpha} - V^\alpha p_{;\alpha} = 2\mu e^{\alpha\beta} e_{\alpha\beta} + f\theta^2 - (Q^\beta_{;\beta} + c^2 A_\alpha Q^\alpha)$$

$$\rho_0 T \frac{Ds}{D\tau} = \rho_0 \left[\frac{D\epsilon}{D\tau} + p \frac{D(1/\rho_0)}{D\tau} \right] = 2\mu e^{\alpha\beta} e_{\alpha\beta} + f\theta^2 - (Q^\beta_{;\beta} + c^2 Q^\alpha A_\alpha)$$

$$\rho_0 \frac{Ds}{D\tau} + \left(\frac{Q^\alpha}{T} \right)_{;\alpha} \geq 0$$

$$2\mu \frac{e^{\alpha\beta} e_{\alpha\beta}}{T} + \frac{f\theta^2}{T} - (Q^\beta_{;\beta} + c^2 Q^\alpha A_\alpha) T^{-1} + \left(\frac{Q^\alpha}{T} \right)_{;\alpha} \geq 0$$

$$-\frac{Q^\beta_{;\beta}}{T} + \left(\frac{Q^\beta}{T} \right)_{;\beta} - \frac{c^2 Q^\alpha A_\alpha}{T}$$

$$= -\frac{Q^\beta_{;\beta}}{T} + \frac{Q^\beta_{;\beta}}{T} + Q^\beta \left(\frac{-1}{T^2} \right) T_{;\beta} - \frac{c^2 Q^\alpha A_\alpha}{T}$$

$$= -\frac{Q^\alpha}{T^2} T_{;\alpha} - \frac{c^2 Q^\alpha A_\alpha}{T} = -\frac{Q^\alpha}{T^2} \left(\frac{T_{;\alpha}}{T} + c^2 A_\alpha \right)$$

$$\left\{ \frac{D\rho_{00}}{D\tau} + \theta [\rho_{00} + c^2(p - \rho\theta)] \right\} V^\alpha + [\rho_{00} + c^2(p - \rho\theta)] A^\alpha$$

$$+ P^{\alpha\beta} (p - \rho\theta)_{,\beta} - 2\mu e^{\alpha\beta}_{;\beta} + c^{-2} \left[DQ^\alpha/D\tau + V^\alpha Q^\beta_{;\beta} + \frac{4}{3} \theta Q^\alpha + Q_\beta (e^{\alpha\beta} + \omega^{\alpha\beta}) \right] = F^\alpha$$

$$V_\alpha V^\alpha \left\{ \frac{D\rho_{00}}{D\tau} + \theta [\rho_{00} + c^2(p - \rho\theta)] \right\} + \cancel{[\rho_{00} + c^2(p - \rho\theta)]} V_\alpha A^\alpha$$

$$+ (p - \rho\theta)_{,\beta} \cancel{V_\alpha P^{\alpha\beta}} - 2\mu V_\alpha e^{\alpha\beta}_{;\beta} + c^{-2} \left[\cancel{V_\alpha DQ^\alpha/D\tau} + V_\alpha V^\alpha Q^\beta_{;\beta} + \frac{4}{3} \theta V_\alpha Q^\alpha + V_\alpha Q^\beta (e^{\alpha\beta} + \omega^{\alpha\beta}) \right] = V_\alpha F^\alpha = 0$$

$$(-c^2) \left\{ \frac{D\rho_{00}}{D\tau} + \theta [\rho_{00} + c^2(p - \rho\theta)] \right\} + 0 + 0 - 2\mu V_\alpha e^{\alpha\beta}_{;\beta}$$

$$+ c^2 \left[V_\alpha DQ^\alpha/D\tau + \underbrace{(V_\alpha V^\alpha)}_{-c^2} Q^\beta_{;\beta} + 0 + Q^\beta (V_\alpha e^{\alpha\beta} + V_\alpha \omega^{\alpha\beta}) \right]$$

$$0 = -c^2 \left\{ \frac{D\rho_{00}}{D\tau} + \theta [\rho_{00} + c^2(p - \rho\theta)] - 2\mu V_\alpha e^{\alpha\beta}_{;\beta} + c^2 V_\alpha DQ^\alpha/D\tau + c^2(c^2) Q^\beta_{;\beta} \right\}$$

~~# c^2 Q^\beta_{;\beta}~~

$$= -c^2 \left\{ \frac{D\rho_{00}}{D\tau} + \theta \rho_{000} - c^2 \rho \theta^2 \right\} - 2\mu V_\alpha e^{\alpha\beta}_{;\beta} + c^2 V_\alpha DQ^\alpha/D\tau - Q^\beta_{;\beta}$$

$$= -c^2 \left(\frac{D\rho_{00}}{D\tau} + \rho_{000} V^\alpha_{;\alpha} \right) + \rho \theta^2 - 2\mu (-e^{\alpha\beta} e_{\alpha\beta}) - c^2 Q^\alpha A_\alpha - Q^\beta_{;\beta}$$

$$= -c^2 \left[\rho_{00,\alpha} + \rho_{000} V^\alpha_{;\alpha} \right] + \rho \theta^2 + 2\mu e^{\alpha\beta} e_{\alpha\beta} - c^2 Q^\alpha A_\alpha - Q^\beta_{;\beta}$$

$$-c^2 \left[(\rho_{000} V^\alpha)_{;\alpha} - c^2 V^\alpha_{;\alpha} \rho \right] + \rho \theta^2 + 2\mu e^{\alpha\beta} e_{\alpha\beta} - c^2 A_\alpha Q^\alpha - Q^\beta_{;\beta}$$

$$P_{\alpha\gamma} V^\gamma = [g_{\alpha\gamma} + c^{-2} V_\alpha V_\gamma] V^\gamma$$

$$= [V_\alpha + c^{-2} V_\alpha (-c^2)] = 0$$

$$P_{\alpha\gamma} M^{\gamma\beta}{}_{;\beta} = P_{\alpha\gamma} V^\gamma \left\{ \frac{Dp_{00}}{Dc} + \theta [p_{00} c^{-2} (p - \beta_0)] \right\}$$

$$+ P_{\alpha\gamma} A^\gamma [p_{00} + c^{-2} (p - \beta_0)] + P_{\alpha\gamma} P^{\gamma\beta} (p - \beta_0)_{;\beta}$$

$$- 2\mu P_{\alpha\gamma} e^{\gamma\beta}{}_{;\beta} + c^{-2} P_{\alpha\gamma} \left[(DQ^\gamma/Dc) + \frac{4}{3} \theta Q^\gamma + Q_\beta (e^{\gamma\beta} + \omega^{\gamma\beta}) \right]$$

$$(P_{\alpha\gamma} P^{\gamma\beta}) = (g_{\alpha\gamma} + c^{-2} V_\alpha V_\gamma) (g^{\gamma\beta} + c^{-2} V^\gamma V^\beta)$$

$$= g_{\alpha\gamma} g^{\gamma\beta} + c^{-2} (g_{\alpha\gamma} V^\gamma V^\beta + g^{\gamma\beta} V_\alpha V_\gamma) + c^{-4} V_\alpha V_\gamma V^\gamma V^\beta$$

$$= \delta_\alpha^\beta + c^{-2} (V_\alpha V^\beta + V^\beta V_\alpha) + c^{-4} V_\alpha V^\beta (-c^2)$$

$$= \delta_\alpha^\beta + c^{-2} (V_\alpha V^\beta) = P_\alpha^\beta$$

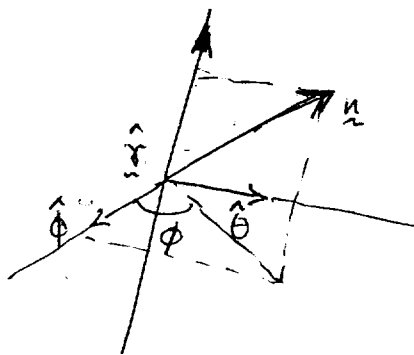
$$P_{\alpha\gamma} M^{\alpha\beta}{}_{;\beta} = A_\alpha [p_{00} + c^{-2} (p - \beta_0)] + P_\alpha^\beta (p - \beta_0)_{;\beta} - 2\mu P_{\alpha\gamma} e^{\gamma\beta}{}_{;\beta}$$

$$+ c^{-2} P_{\alpha\gamma} \left[DQ^\gamma/Dc + \frac{4}{3} \theta Q^\gamma + Q_\beta (e^{\gamma\beta} + \omega^{\gamma\beta}) \right]$$

$$P_{\alpha\gamma} Q^\gamma = [g_{\alpha\gamma} + c^{-2} V_\alpha V_\gamma] Q^\gamma = Q_\alpha$$

Section 89.1 The Photon Four-Momentum

89.4 $ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$
 $= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$



hence
 $n_r = \mu$
 $n_\theta = (1-\mu^2)^{1/2} \cos \phi$
 $n_\phi = (1-\mu^2)^{1/2} \sin \phi$

hence
 $n^{(r)} = \mu$
 $n^{(\theta)} = (1-\mu^2)^{1/2} \cos \phi / r$
 $n^{(\phi)} = (1-\mu^2)^{1/2} \sin \phi / r \sin \theta$

and $M^1 = (h\nu/c) n^{(r)}$; $M^2 = (h\nu/c) n^{(\theta)}$; $M^3 = (h\nu/c) n^{(\phi)}$

(1) (89.5) + (89.6) [next page] to obtain (89.10) and (89.11)

$$\begin{pmatrix} m_0^0 \\ m_0^1 \\ m_0^2 \\ m_0^3 \end{pmatrix} = \frac{h\nu_0}{c} \begin{pmatrix} 1 \\ \mu_0 \\ (1-\mu_0^2)^{1/2} \cos \phi_0 \\ (1-\mu_0^2)^{1/2} \sin \phi_0 \end{pmatrix} = \begin{pmatrix} \gamma - \gamma v_z/c & 0 & 0 \\ -\gamma v_z/c [\frac{1}{\gamma} + (\gamma-1)\beta^2 \beta^2] & 0 & 0 \\ 0 & 0 [\frac{1}{\gamma} + (\gamma-1)\beta^2 \beta^2] & 0 \\ 0 & 0 & 0 [\frac{1}{\gamma} + (\gamma-1)\beta^2 \beta^2] \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ (1-\mu^2)^{1/2} \cos \phi \\ (1-\mu^2)^{1/2} \sin \phi \end{pmatrix} \frac{h\nu}{c}$$

(a) $\nu_0 = \gamma \nu - \gamma v_z \frac{\nu}{c} \mu = \gamma \nu (\gamma - \beta \mu)$
 $\frac{h\nu_0}{c} \mu_0 = -\gamma \beta \frac{h\nu}{c} + [\frac{1}{\gamma} + (\gamma-1)\beta^2 \beta^2] \mu \frac{h\nu}{c}$
 $\Rightarrow \nu_0 \mu_0 = -\gamma \beta \nu + \mu \nu (\gamma - \beta) + \mu \nu = \gamma \nu (\mu - \beta)$
 hence $\mu_0 \cdot \gamma \nu (1 - \beta \mu) = \gamma \nu (\mu - \beta)$

(b) $\mu_0 = (\mu - \beta) / (1 - \beta \mu)$
 $(1-\mu_0^2)^{1/2} \cos \phi_0 \frac{h\nu_0}{c} = [\frac{1}{\gamma} + (\gamma-1)\beta^2 \beta^2] (1-\mu^2)^{1/2} \cos \phi \frac{h\nu}{c}$
 $(1-\mu_0^2)^{1/2} \cos \phi_0 \nu_0 = (1-\mu^2)^{1/2} \cos \phi \nu$
 $\gamma \nu (1 - \beta \mu) (1-\mu_0^2)^{1/2} \cos \phi_0 = \nu (1-\mu^2)^{1/2} \cos \phi$
 $(1-\mu_0^2)^{1/2} \cos \phi_0 = (1-\mu^2)^{1/2} \cos \phi / [\gamma (1 - \beta \mu)]$

Similarly $(1-\mu_0^2)^{1/2} \sin \phi_0 = (1-\mu^2)^{1/2} \sin \phi / [\gamma (1 - \beta \mu)]$

(c) + (d) From which: $\phi_0 = \phi$ $(1-\mu_0^2)^{1/2} = (1-\mu^2)^{1/2} / [\gamma (1 - \beta \mu)]$

$$\begin{pmatrix} \gamma_0 \\ \gamma_0 \underline{n}_0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \underline{\beta} \\ -\gamma \underline{\beta} & \underline{I} + (\gamma-1) \underline{\beta} \underline{\beta} \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \underline{n} \end{pmatrix}$$

(89.5)

$$\gamma_0 = \gamma \gamma - \gamma \beta \gamma \underline{n} = \gamma \gamma (1 - \beta \underline{n}) = \gamma \gamma (1 - \underline{n} \cdot \underline{v} / c)$$

$$\begin{aligned} \gamma_0 \underline{n}_0 &= -\gamma \underline{\beta} \gamma + [\underline{I} + (\gamma-1) \underline{\beta} \underline{\beta}] \cdot \gamma \underline{n} = \gamma \underline{n} - \gamma \underline{v} \underline{v} / c + \gamma (\gamma-1) (\underline{v} \cdot \underline{n}) \underline{v} / v^2 \\ &= \gamma \underline{n} - \gamma \underline{v} \underline{v} / c [1 - (\underline{n} \cdot \underline{v} / v^2) (\gamma-1) / \gamma] \end{aligned}$$

$$\begin{aligned} \text{now } (\gamma-1) / \gamma &= (\gamma-1)(\gamma+1) / \gamma(\gamma+1) = (\gamma^2-1) / [\gamma(\gamma+1)] = [1 - v^2/c^2]^{-1} \frac{1}{\gamma(\gamma+1)} \\ &= (v^2/c^2) \frac{1}{(1-v^2/c^2)} \frac{1}{\gamma(\gamma+1)} = \frac{\gamma^2}{\gamma(\gamma+1)} \frac{v^2}{c^2} = \frac{\gamma}{\gamma+1} \frac{v^2}{c^2} \end{aligned}$$

hence

(89.6)

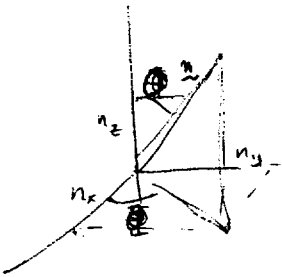
$$\gamma_0 \underline{n}_0 = \gamma \underline{n} - \gamma \underline{v} \underline{v} / c \left[1 - \frac{\underline{n} \cdot \underline{v} / c}{v^2} \frac{\gamma}{\gamma+1} \frac{v^2}{c^2} \right] = \gamma \underline{n} - \gamma \underline{v} \underline{v} / c \left[1 - \frac{\underline{n} \cdot \underline{v}}{c} \frac{\gamma}{\gamma+1} \right]$$

Inverse:

$$\begin{pmatrix} \gamma \\ \gamma \underline{n} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \underline{\beta} \\ \gamma \underline{\beta} & \underline{I} + (\gamma-1) \underline{\beta} \underline{\beta} \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_0 \underline{n}_0 \end{pmatrix}$$

$$\gamma = \gamma \gamma_0 + \gamma \beta \gamma_0 \underline{n}_0 = \gamma \gamma_0 [1 + \underline{n}_0 \cdot \underline{v} / c]$$

$$\begin{aligned} \gamma \underline{n} &= \gamma \underline{\beta} \gamma_0 + [\underline{I} + (\gamma-1) \underline{\beta} \underline{\beta}] \gamma_0 \underline{n}_0 = \gamma \underline{\beta} \gamma_0 + \gamma_0 \underline{n}_0 + (\gamma-1) (\underline{v} \cdot \underline{n}_0 / v^2) \underline{v} \\ &= \gamma_0 \underline{n}_0 + \gamma \left[1 + (\gamma-1) \frac{\underline{v} \cdot \underline{n}_0}{v^2} \right] \frac{\underline{v}}{c} = \gamma_0 \underline{n}_0 + \gamma \frac{\underline{v}}{c} \left[1 + \frac{\underline{v} \cdot \underline{n}_0}{v^2} \frac{\gamma}{\gamma+1} \right] \end{aligned}$$



$$\begin{array}{r} .005 \\ 50 \\ \hline 0.250 \end{array}$$

$$00.125.36000000$$

$$125.36$$

Section 90: Transformation Laws for the Specific Intensity, Opacity + Emissivity

$$N = \left(\frac{I}{h\nu}\right) (d\omega d\nu) (dS \cos \theta dt) = \left(\frac{I}{h\nu}\right) \left(\frac{\nu d\nu d\omega}{\nu}\right) (dS dt) \mu$$

$$N_0 = \left(\frac{I_0}{h\nu_0}\right) (d\omega_0 d\nu_0) (dS \cos \theta_0 dt_0 + \beta dS dt_0)$$

$$= \left(\frac{I_0}{h\nu_0}\right) \left(\frac{\nu_0 d\nu_0 d\omega_0}{\nu_0}\right) (dS dt_0) (\mu_0 + \beta)$$

Now since $N = N_0$:

$$\left(\frac{I_0}{h\nu_0}\right) \frac{dt_0}{\nu_0} (\mu_0 + \beta) \left[dS \cancel{\nu_0 d\nu_0 d\omega_0} \right] = \left(\frac{I}{h\nu}\right) \frac{dt}{\nu} \mu \left[dS \cancel{\nu d\nu d\omega} \right]$$

$$dt_0 = \gamma dt$$

$$\mu = (\mu_0 + \beta) / (1 + \beta \mu_0) \text{ but } (1 + \beta \mu_0) = \gamma / \gamma \nu_0 \text{ hence}$$

$$\mu = (\mu_0 + \beta) (\gamma \nu_0 / \nu)$$

Therefore

$$\frac{I_0}{\nu_0^2} \cancel{\gamma dt} (\cancel{\mu_0 + \beta}) = \frac{I}{\nu^2} dt (\cancel{\mu_0 + \beta}) \left(\frac{\gamma \nu_0}{\nu}\right)$$

$$(90.3) \Rightarrow I(\mu, \nu) = (\nu / \nu_0)^3 I_0(\mu_0, \nu_0)$$

$$\text{or } (I / \nu^3) = (I_0 / \nu_0^3) = \text{Lorentz invariant}$$

$$\eta_0 d\omega_0 d\nu_0 dV_0 dt_0 / h\nu_0 = \eta d\omega d\nu dV dt / h\nu$$

$dV dt$ is an invariant; and $\nu d\nu d\omega$ is invariant, hence

$$\eta_0 \left(\frac{\nu_0 d\nu_0 d\omega_0}{\nu_0}\right) \frac{1}{\nu_0} = \eta \left(\frac{\nu d\nu d\omega}{\nu}\right) \frac{1}{\nu}$$

$$(90.6) \Rightarrow \eta_0 / \nu_0^2 = \eta / \nu^2$$

$$\chi_0 I_0 (d\nu_0 d\omega_0 dV_0 dt_0) / \nu_0 = \chi I (d\nu d\omega dV dt) / \nu$$

$$\Rightarrow \chi_0 I_0 (\nu_0 d\nu_0 d\omega_0 / \nu_0) (1/\nu_0) = \chi I (\nu d\nu d\omega / \nu) (1/\nu)$$

$$\chi_0 I_0 / \nu_0^2 = \chi I / \nu^2$$

But $I = (\nu^3 / \nu_0^3) I_0$ hence

$$\chi_0 I_0 / \nu_0^2 = \chi I_0 (\nu^3 / \nu_0^3) / \nu^2$$

$$\chi_0 \nu_0 = \chi \nu$$

$$\frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I}{\partial \mu} = \eta - \chi I$$

$$89.7: v = \gamma v_0 (1 + \frac{u_0 v}{c})$$

$$90.3: I = (v/v_0)^3 I_0$$

$$90.6: \eta = (v/v_0)^2 \eta_0$$

$$90.8: \chi = (v_0/v) \chi_0$$

$$\Rightarrow \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{v^3}{v_0^3} I_0 \right) + \mu \frac{\partial}{\partial r} \left(\frac{v^3}{v_0^3} I_0 \right) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} \left(\frac{v^3}{v_0^3} I_0 \right) = \frac{v^2}{v_0^2} \eta_0 - \left(\frac{v_0}{v} \right) \chi_0 \left(\frac{v^3}{v_0^3} \right) I_0$$

$$= \frac{v^3}{v_0^3} \left\{ \frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_0}{\partial \mu} \right\} - 3 \frac{v^3}{v_0^3} \left\{ \frac{I_0}{c} \frac{\partial v_0}{\partial t} + \mu I_0 \frac{\partial v_0}{\partial r} + \frac{(1-\mu^2)}{r} I_0 \frac{\partial v_0}{\partial \mu} \right\}$$

$$= \frac{v^2}{v_0^2} [\eta_0 - \chi_0 I_0]$$

$$\Rightarrow \frac{v}{v_0} \left\{ \frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_0}{\partial \mu} \right\} - 3 \frac{v}{v_0^2} I_0 \left\{ \frac{1}{c} \frac{\partial v_0}{\partial t} + \mu \frac{\partial v_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial v_0}{\partial \mu} \right\} = \eta_0 - \chi_0 I_0$$

now

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} \Big|_{r, \mu, v_0} + \frac{\partial \mu_0}{\partial t} \Big|_{r, \mu} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial t} \Big|_{r, \mu} \frac{\partial}{\partial v_0}$$

hence

$$\frac{v}{v_0} \frac{1}{c} \frac{\partial I_0}{\partial t} \rightarrow \frac{v}{v_0} \frac{1}{c} \left[\frac{\partial I_0}{\partial t} + \frac{\partial \mu_0}{\partial t} \frac{\partial I_0}{\partial \mu_0} + \frac{\partial v_0}{\partial t} \frac{\partial I_0}{\partial v_0} \right]; \frac{1}{c} \frac{\partial v_0}{\partial t} \rightarrow \frac{1}{c} \left[\frac{\partial v_0}{\partial t} + \right.$$

Section 91: The Radiation Stress-Energy Tensor and Four-Force Vector

$$R^{\alpha\beta} = c^{-1} \int dv \int d\omega I n^\alpha n^\beta \quad n^0 = 1$$

$$\begin{aligned} & \int dv \int d\omega (I/v^3) \left(\frac{h\nu}{c} n^\alpha\right) \left(\frac{h\nu}{c} n^\beta\right) = \frac{h^2}{c^2} \int dv \int d\omega I n^\alpha n^\beta \\ & = \text{product of } M^\alpha M^\beta; (I/v^3); v dv d\omega \\ & = (h^2/c) R^{\alpha\beta} \end{aligned}$$

$$R^{\alpha\beta} = c^2 \int_{\mathcal{R}} M^\alpha M^\beta d^3M / \tilde{e} \quad \tilde{e} = h\nu \quad M^\alpha = \frac{h\nu}{c} n^\alpha \text{ etc.}$$

From (63.4) $f_R = \frac{c^2}{h^4 v^3} \mathbb{I}$

Also $d^3M = p^2 dp d\omega$, $p = \frac{h\nu}{c} \Rightarrow d^3M = \frac{h^3}{c^3} v^2 dv d\omega$
hence

$$\begin{aligned} R^{\alpha\beta} &= c^2 \int \left[\frac{c^2}{h^4 v^3} \mathbb{I} \right] \left(\frac{h\nu}{c} n^\alpha\right) \left(\frac{h\nu}{c} n^\beta\right) \frac{h^3}{c^3} v^2 dv d\omega / h\nu \\ &= \frac{c^4 h^5}{c^5 h^5} \int \frac{\mathbb{I}}{v^3} v^2 n^\alpha n^\beta \cdot \frac{v^2 dv d\omega}{v} \end{aligned}$$

(91.6)

$$\boxed{R^{\alpha\beta} = c^{-1} \int \mathbb{I} n^\alpha n^\beta dv d\omega}$$

(91.7) From (64.4) $E_\nu = c^{-1} \int I d\omega$
(64.5) $E = \int dv E_\nu = c^{-1} \int dv \int I d\omega = R^{00}$

(65.2) $\underline{F}_\nu = \int \mathbb{I} \underline{n} d\omega \quad F_\nu^i = \int \mathbb{I} n^i d\omega$

(65.4) $F^i = \int dv F_\nu = \int dv \int d\omega \mathbb{I} n^i \quad i=1,2,3$

hence

$$R^{0i} = c^{-1} F^i$$

$$R^{i0} = c^{-1} F^i$$

Finally $P_{\nu}^{ij} = c^{-1} \int \mathbb{I} n^i n^j d\omega \quad (66.3)$

$$P^{ij} = c^{-1} \int dv \int \mathbb{I} n^i n^j d\omega = R^{ij} \quad i, j = 1, 2, 3$$

Relate E to E^0 :

$$E = c' \int \mathbf{I} \, v \, d\omega = c^{-1} \int \frac{\mathbf{I}}{v^3} (v \, v \, d\omega) \left(\frac{h\nu}{c}\right) \left(\frac{h\nu}{c}\right) \frac{c^2}{h^2}$$

$$= c^{-1} \int \left[\frac{\mathbf{I}^0}{v_0^3} (v_0 \, v_0 \, d\omega_0) \right] v_0^2 \gamma^2 [1 + \beta \cdot \underline{n}]^2$$

$$= c^{-1} \int \mathbf{I}_0 \, dv_0 \, d\omega_0 \cdot \gamma^2 [1 + \beta \cdot \underline{n}]^2$$

$$E = c^{-1} \gamma^2 \left\{ \int \mathbf{I}_0 \, dv_0 \, d\omega_0 + 2\beta \cdot \int \mathbf{I}_0 \, dv_0 \, d\omega_0 \underline{n}_0 + \int \mathbf{I}_0 \, dv_0 \, d\omega_0 \beta \cdot \underline{n}_0 \beta \cdot \underline{n}_0 \right\}$$

(9.1.10)

$$E = \gamma^2 \left\{ E_0 + 2c^{-2} v_i F_0^i + c^{-2} v_i v_j P_0^{ij} \right\}$$

Lorentz transform of F :

$$F^i = \int \mathbf{I} \, n^i \, dv \, d\omega = \int \left[\frac{\mathbf{I}}{v^3} (v \, v \, d\omega) \right] v^2 n^i$$

$$= \int \left[\frac{\mathbf{I}_0}{v_0^3} v_0 \, dv_0 \, d\omega_0 \right] v_0 \cdot \gamma \cdot [1 + \beta \cdot \underline{n}] v_0 \left\{ n_0^i + \gamma v_0^i / c [1 + \frac{\gamma \underline{n}_0 \cdot \underline{v}}{c(\gamma+1)}] \right\}$$

$$= \gamma \int \mathbf{I}_0 \, dv_0 \, d\omega_0 [1 + \beta \cdot \underline{n}_0] \left[n_0^i + \gamma v_0^i / c + \gamma^2 n_0^j v_j v^i / c^2 (\gamma+1) \right]$$

$$= \gamma \int \mathbf{I}_0 \, dv_0 \, d\omega_0 (1 + v_k n_0^k / c) \left[n_0^i + \gamma v_0^i / c + \gamma^2 n_0^j v_j v^i / c^2 (\gamma+1) \right]$$

$$= \gamma \int \mathbf{I}_0 \, dv_0 \, d\omega_0 \left\{ n_0^i + \gamma v_0^i / c + \gamma^2 n_0^j v_j v^i / c^2 (\gamma+1) + v_k n_0^i n_0^k + \gamma v_0^i v_k n_0^k / c \right.$$

$$\left. + \gamma^2 n_0^j n_0^k v_j v_k / c^2 (\gamma+1) \right\}$$

$$= (\gamma^2 v_0^i / c) \int \mathbf{I}_0 \, dv_0 \, d\omega_0 + \gamma \int \mathbf{I}_0 \, dv_0 \, d\omega_0 n_0^i + (\gamma^3 v_0^i v_j / c^2 (\gamma+1)) \int \mathbf{I}_0 \, dv_0 \, d\omega_0 n_0^j$$

$$+ \gamma v_k \int \mathbf{I}_0 \, dv_0 \, d\omega_0 n_0^i n_0^k + \gamma^2 v_0^i v_k / c^2 \int \mathbf{I}_0 \, dv_0 \, d\omega_0 n_0^k + (\gamma^3 v_0^i v_j v_k / c^3 (\gamma+1)) \int \mathbf{I}_0 \, dv_0 \, d\omega_0 n_0^j n_0^k$$

$$= \gamma^2 v_0^i E_0 + \gamma F_0^i + [\gamma^3 / (\gamma+1)] c^{-2} v_0^i v_j F_0^j + \gamma v_k P_0^{ik} + \gamma^2 c^{-2} v_0^i v_k F_0^k + \frac{\gamma^3}{(\gamma+1)} v_0^i c^{-2} v_j v_k P_0^{jk}$$

$$\left\{ F_0^i + \gamma v_0^i E_0 + v_k P_0^{ik} + c^{-2} v_0^i \gamma \left[v_k F_0^k + \frac{\gamma}{(\gamma+1)} v_j F_0^j + \frac{\gamma}{(\gamma+1)} v_j v_k P_0^{jk} \right] \right\}$$

$$\frac{\gamma}{\gamma+1} v_k F_0^k + \frac{(\gamma+1)}{(\gamma+1)} v_k F_0^k = \frac{2\gamma+1}{\gamma+1} v_k F_0^k$$

(9.1.11) Hence $F^i = \gamma \left\{ F_0^i + \gamma v^i E_0 + v_k P_0^{ik} + \frac{\gamma}{c^2(\gamma+1)} \left[(2\gamma+1) v_k F_0^k + \gamma v_j v_k P_0^{jk} \right] \right\}$

Lorentz transform of P^{ij}

$$P^{ij} = \int I_c dv dw n^i n^j = \int \frac{I_0}{v_0^3} (\gamma_0 dv_0 dw_0) \cdot (v n^i) (v n^j)$$

$$v n^i = \gamma_0 \left\{ n_0^i + (\gamma v^i/c) \left[1 + \gamma n_0^k v_k^i / c(\gamma+1) \right] \right\}$$

$$v n^j = \gamma_0 \left\{ n_0^j + \gamma v^j/c \left[1 + \gamma n_0^l v_l^j / c(\gamma+1) \right] \right\}$$

hence

$$P^{ij} = \int I_0 dv_0 dw_0 \left\{ n_0^i + \frac{\gamma v^i}{c} \left[1 + \frac{\gamma n_0^k v_k^i}{c(\gamma+1)} \right] \right\} \left\{ n_0^j + \frac{\gamma v^j}{c} \left[1 + \frac{\gamma n_0^l v_l^j}{c(\gamma+1)} \right] \right\}$$

$$= \int I_0 dv_0 dw_0 \left\{ n_0^i + \frac{\gamma v^i}{c} + \frac{\gamma^2 n_0^k v^i v_k}{c^2(\gamma+1)} \right\} \left\{ n_0^j + \frac{\gamma v^j}{c} + \frac{\gamma^2 n_0^l v^j v_l}{c^2(\gamma+1)} \right\}$$

$$= \int I_0 dv_0 dw_0 \left\{ n_0^i n_0^j + n_0^i \frac{\gamma v^j}{c} + n_0^j \frac{\gamma v^i}{c} + \frac{\gamma^2 n_0^i n_0^l v^j v_l}{c^2(\gamma+1)} + \frac{\gamma^2 v^i v^j}{c^2} \right. \\ \left. + \frac{\gamma^3 n_0^l v^i v^j v_l}{c^3(\gamma+1)} + \frac{\gamma^2 n_0^k n_0^j v^i v_k}{c^2(\gamma+1)} + \frac{\gamma^3 n_0^k v^i v^j v_k}{c^3(\gamma+1)} + \frac{\gamma^4 n_0^k n_0^l v^i v_k v_l v_j}{c^4(\gamma+1)^2} \right\}$$

$$= P_0^{ij} + \gamma c^{-2} v^j F_0^i + \gamma c^{-2} v^i F_0^j + \frac{\gamma^2}{c^2(\gamma+1)} v^j v_l P_0^{il} + \frac{\gamma^2}{c^2} v^i v^j E_0 \\ + \frac{\gamma^3}{c^3(\gamma+1)} v^i v^j v_l F_0^l + \frac{\gamma^2}{c^2(\gamma+1)} v^i v_k P_0^{ik} + \frac{\gamma^3}{c^3(\gamma+1)^2} v^i v^j v_k F_0^k + \frac{\gamma^4}{c^4(\gamma+1)^2} v^i v^j v_k v_l P_0^{kl}$$

$$= P_0^{ij} + \frac{\gamma}{c^2} \left[v^j F_0^i + v^i F_0^j + \gamma v^i v^j E_0 \right]$$

$$+ \frac{\gamma^2}{c^2(\gamma+1)} \left[v^j v_l P_0^{il} + \frac{\gamma}{c^2} v^i v^j v_l F_0^l + v^i v_k P_0^{kj} + \frac{\gamma}{c^2} v^i v^j v_k F_0^k + \frac{\gamma^2 v^i v^j v_k v_l}{c^2(\gamma+1)} \right]$$

(a11) $P^{ij} = P_0^{ij} + \gamma c^{-2} \left[v^i F_0^j + v^j F_0^i + \gamma v^i v^j E_0 \right]$

$$+ \frac{\gamma^2}{c^2(\gamma+1)} \left[v^j v_k P_0^{ik} + v^i v_k P_0^{kj} + 2\gamma c^{-2} v^i v^j v_k F_0^k \right] + \frac{\gamma^4}{c^4(\gamma+1)^2} v^i v^j v_k v_l P_0^{kl}$$

One dimensional flow:

$$E: \quad E = \gamma^2 [E_0 + 2c^{-1} v_i F_0^i + c^{-2} v_i v_j P_0^{ij}]$$

only one surviving velocity component, hence sums over $i+j$ become: $v_i F_0^i = v F_0$ $v_i v_j P_0^{ij} = v^2 P_0$ and $v^2 c^{-2} P_0 = \beta^2 P_0$

$$(9.13) \quad E = \gamma^2 [E_0 + 2\beta F_0 + \beta^2 P_0]$$

$$F: \quad F^i = \gamma [F_0^i + \gamma v^i E_0 + v_k P_0^{ik} + (\gamma v^i / c^2 (\gamma+1)) [(2\gamma+1) v_k F_0^k + \gamma v_j v_k P_0^{jk}]]$$

$$= \{ \gamma^2 v E_0 + F_0 [\gamma + \gamma^2 \beta^2 (2\gamma+1) / (\gamma+1)] + P_0 v \gamma [1 + \gamma^2 \beta^2 / (\gamma+1)] \}$$

Now $\beta^2 = (\gamma^2 - 1) / \gamma^2$, hence $\gamma^2 \beta^2 / (\gamma+1) = \gamma^2 (\gamma+1)(\gamma-1) / \gamma^2 (\gamma+1) = (\gamma-1)$

Therefore the coeff of F_0 is:

$$\gamma + \gamma^2 \beta^2 (2\gamma+1) / (\gamma+1) = \gamma + (2\gamma+1)(\gamma-1) = \gamma + 2\gamma^2 - \gamma - 1 = 2\gamma^2 - 1 = \frac{2}{1-\beta^2} - 1$$

$$= [2 - (1-\beta^2)] / (1-\beta^2) = (1+\beta^2) / (1-\beta^2) = \gamma^2 (1+\beta^2)$$

and the coeff of $\gamma P_0 v$:

$$1 + \gamma^2 \beta^2 / (\gamma+1) = 1 + (\gamma-1) = \gamma$$

hence

$$(9.14) \quad F = \gamma^2 v E_0 + \gamma^2 (1+\beta^2) F_0 + \gamma^2 v P_0$$

$$\boxed{F = \gamma^2 [v E_0 + (1+\beta^2) F_0 + v P_0]}$$

$$P: \quad P^{ij} = P_0^{ij} + \gamma c^{-2} [v^i F_0^j + v^j F_0^i + \gamma v^i v^j E_0]$$

$$+ \frac{\gamma^2}{c^2 (\gamma+1)} [v^i v_k P_0^{jk} + v^j v_k P_0^{ki} + 2\gamma c^{-2} v^i v^j v_k F_0^k] + \left(\frac{\gamma^2}{c^2 (\gamma+1)}\right)^2 v^i v^j v_k v_l P_0^{kl}$$

$$= P_0 + 2\gamma c^{-1} \beta F_0 + \gamma^2 \beta^2 E_0 + \frac{\gamma^2}{(\gamma+1)} [2\beta^2 P_0 + 2\gamma c^{-1} \beta^3 F_0] + \left[\frac{\gamma^2}{(\gamma+1)}\right]^2 \beta^4 P_0$$

$$= \gamma^2 \beta^2 E_0 + P_0 \left\{ 1 + \frac{2\beta^2 \gamma^2}{(\gamma+1)} + \left[\frac{\beta^2 \gamma^2}{\gamma+1}\right]^2 \right\} + F_0 \cdot 2\gamma c^{-1} \beta \left[1 + \frac{\gamma^2 \beta^2}{(\gamma+1)} \right]$$

$$= \gamma^2 \beta^2 E_0 + P_0 \{ 1 + 2(\gamma-1) + (\gamma-1)^2 \} + 2\gamma c^{-1} \beta F_0 \{ 1 + (\gamma-1) \}$$

$$= \gamma^2 \beta^2 E_0 + P_0 \{ 1 + 2\gamma - 2 + \gamma^2 - 2\gamma + 1 \} + 2\gamma c^{-1} \beta F_0$$

$$(9.15) \quad = \gamma^2 \{ P_0 + \beta^2 E_0 + 2v F_0 \}$$

To first order in β this becomes

$$(9.1.16a) \quad E = E_0 + 2\beta \bar{c}' F_0 = E_0 + 2v c^{-2} F_0$$

$$(9.1.17a) \quad F = F_0 + v(E_0 + P_0)$$

$$(9.1.18a) \quad P = P_0 + 2\beta \bar{c}' F_0 = P_0 + 2v c^{-2} F_0$$

Then using $E_0 = 4\pi \bar{c}' J_0$; $E = 4\pi \bar{c}' J$
 $F_0 = 4\pi H_0$; $F = 4\pi H$
 $P_0 = 4\pi \bar{c}' K_0$; $P = 4\pi \bar{c}' K$

we have

$$(9.1.16b) \quad J = J_0 + 2\beta F_0$$

$$(9.1.17b) \quad H = H_0 + \beta(J_0 + K_0)$$

$$(9.1.18b) \quad K = K_0 + 2\beta H_0$$

$$\begin{aligned} G^0 &= \bar{c}' \int dv dw [xI - \eta] \\ &= \bar{c}' \int (v dv dw) [v^{-2}(xI - \eta)] v \\ &= \bar{c}' \int (v_0 dv_0 dw_0) v_0^{-2} (x_0 I_0 - \eta_0) \gamma v_0 (1 + \beta \cdot \underline{n}) \\ &= \bar{c}' \int dv_0 dw_0 (x_0 I_0 - \eta_0) + \gamma \beta \cdot \left[\bar{c}' \int dv_0 dw_0 [I x - \eta] \underline{n} \right] \end{aligned}$$

$$(9.1.22a) \quad G^0 = G_0^0 + \gamma \beta \cdot G_0$$

$$\begin{aligned} G^1 &= \bar{c}' \int dv dw [xI - \eta] \underline{n} \\ &= \bar{c}' \int (v dv dw) v^{-2} [xI - \eta] (v \underline{n}) \\ &= \bar{c}' \int (v_0 dv_0 dw_0) v_0^{-2} [x_0 I_0 - \eta_0] \gamma v_0 \left[\underline{n}_0 + \gamma \beta \left(1 + \frac{\gamma \underline{n}_0 \cdot \beta}{(\gamma + 1)} \right) \right] \\ &= \bar{c}' \int dv_0 dw_0 [x_0 I_0 - \eta_0] \left[\underline{n}_0 + \gamma \beta + \gamma^2 (\underline{n}_0 \cdot \beta) \beta / (\gamma + 1) \right] \end{aligned}$$

For one-dimensional flow,

$$\begin{aligned} \underline{n}_0 + \gamma \beta + \gamma^2 (\underline{n}_0 \cdot \beta) \beta / (\gamma + 1) &= \underline{n}_0 + \gamma \beta + [\gamma^2 \beta^2 / (\gamma + 1)] \underline{n}_0 \\ &= \underline{n}_0 + (\gamma - 1) \underline{n}_0 + \gamma \beta = \gamma (\beta + \underline{n}_0) \end{aligned}$$

$$\begin{aligned} G^1 &= \bar{c}' \int dv_0 dw_0 [x_0 I_0 - \eta_0] [\gamma \beta + \gamma \underline{n}_0] \\ (9.1.22b) \quad &= \gamma \beta G_0^0 + \gamma G_0^1 \end{aligned}$$

The reverse procedure, with $v_0 = \gamma v (1 - \underline{n} \cdot \underline{\beta})$
 and $v_0 \underline{n}_0 = \gamma [\underline{n} - \gamma \underline{\beta} (1 - \gamma \underline{n} \cdot \underline{\beta} / (\gamma + 1))]]$
 yields for 1-D flow:

$$(91.23a) \quad G_0^0 = \gamma (G^0 - \beta G^1)$$

$$(91.23b) \quad G_0^1 = \gamma (G^1 - \beta G^0)$$

Section 92 : Covariant Form of the Transfer Equation

Generalize the Boltzmann equation

$$\partial f / \partial t + v^i (\partial f / \partial x_i) + \dot{p}^i (\partial f / \partial p_i) = (Df/Dt)_{\text{coll}}$$

to

$$(92.4) \quad \left(\frac{dx^\alpha}{d\lambda} \right) \frac{\partial f}{\partial x^\alpha} + \left(\frac{dp^\alpha}{d\lambda} \right) \left(\frac{\partial f}{\partial p^\alpha} \right) = \left(\frac{Df}{D\lambda} \right)_{\text{coll}} = p^\alpha \left(\frac{\partial f}{\partial x^\alpha} \right) + \dot{p}^\alpha \left(\frac{\partial f}{\partial p^\alpha} \right)$$

$$(92.5) \quad \text{where we define } p^\alpha \equiv (dx^\alpha / d\lambda)$$

$$(92.5) \quad \dot{p}^\alpha \equiv (dp^\alpha / d\lambda)$$

λ is used in (92.4) instead of proper time τ because photon trajectories lie on null cone.

If $M^\alpha = p^\alpha$ and $\epsilon = \text{source}$, $-\alpha f_R = \text{sink}$ then:

$$M^\alpha \frac{\partial f_R}{\partial x^\alpha} + \dot{M}^\alpha \frac{\partial f_R}{\partial M^\alpha} = \epsilon - \alpha f_R \equiv \left(\frac{Df_R}{D\lambda} \right)$$

$$\text{From (63.4): } I = (h^4 v^3 / c^2) f_R \Rightarrow f_R = (I / v^3) (c^2 / h^4) = \int c^2 / h^4$$

$$M^\alpha \frac{c^2}{h^4} \frac{\partial \int}{\partial x^\alpha} + \dot{M}^\alpha \frac{c^2}{h^4} \frac{\partial \int}{\partial M^\alpha} = \epsilon - \alpha \frac{c^2}{h^4} \int$$

or

$$(92.7) \quad M^\alpha \frac{\partial \int}{\partial x^\alpha} + \dot{M}^\alpha \frac{\partial \int}{\partial M^\alpha} = \epsilon \left(\frac{h^4}{c^2} \right) - \alpha \int \equiv \epsilon - \alpha \int$$

In an inertial frame, $\dot{M}^\alpha = 0$ for photons, and $v = \text{constant}$, hence

$$M^\alpha \frac{\partial}{\partial x^\alpha} (I / v^3) = \frac{h v}{c} \frac{n^\alpha}{v^3} \frac{\partial I}{\partial x^\alpha}$$

$$= \frac{h}{c} \frac{1}{v^2} \left[\frac{\partial I}{\partial t} + \underline{n} \cdot \nabla I \right] = \epsilon - \alpha \frac{I}{v^3}$$

$$\frac{1}{\gamma^2} \left[\frac{1}{c} \frac{\partial I}{\partial t} + \underline{n} \cdot \nabla I \right] = \frac{e}{h} e - \frac{c}{h} a \frac{1}{v^2}$$

Right-hand-side must scale as γ^{-2} (like lhs), and we identify

$$(e/h) e = \gamma v / v^2$$

$$(c/h) a = \gamma \chi v$$

Note: in § 92 $c=h=1$ units are used, and then $e = \gamma v / v^2$ $a = \gamma \chi v$
So now we have:

$$\gamma^{-2} [\eta - \chi I] = \gamma^{-2} \left[\frac{1}{c} \frac{\partial I}{\partial t} + (\underline{n} \cdot \nabla) I \right] = \frac{e}{h} M^\alpha \downarrow_{\alpha}$$

But \downarrow = world scalar; \downarrow_{α} = covariant and $M^\alpha \downarrow_{\alpha}$ = invariant.

$$\text{hence } \frac{e}{h} M^\alpha \downarrow_{\alpha} = \frac{e}{h} M'^\alpha \downarrow'_{\alpha} = \gamma^{-2} \left[\frac{1}{c} \frac{\partial I'}{\partial t'} + (\underline{n}' \cdot \nabla') I' \right]$$

and since $\gamma^{-2} [\eta - \chi I]$ is also invariant:

$$[\eta' - \chi' I'] = \frac{1}{c} \frac{\partial I'}{\partial t'} + (\underline{n}' \cdot \nabla') I'$$

(92.11)

Alternate derivation: boost matrix is: $\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = L^{\alpha'}_{\beta}$

$$(92.12) \quad (35.39): B'_\alpha = (\partial x^\beta / \partial x'^\alpha) B_\beta = L^{\beta}_{\alpha'} B_\beta$$

$$\begin{pmatrix} \frac{1}{c} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma \frac{\partial}{\partial t'} - \beta\gamma \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \gamma \frac{\partial}{\partial z'} - \beta\gamma \frac{\partial}{\partial t'} \end{pmatrix}$$

Then

$$\frac{1}{c} \frac{\partial}{\partial t} + \underline{n} \cdot \nabla = \frac{1}{c} \frac{\partial}{\partial t} + n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y} + n_z \frac{\partial}{\partial z}$$

$$= \left(\frac{\gamma}{c} \frac{\partial}{\partial t'} - \gamma\beta \frac{\partial}{\partial z'} \right) + n_x \frac{\partial}{\partial x'} + n_y \frac{\partial}{\partial y'} + n_z \gamma \left(\frac{\partial}{\partial z'} - \beta \frac{\partial}{\partial t'} \right)$$

$$= \left(\frac{\gamma}{c} \frac{\partial}{\partial t'} - \frac{\gamma n_z \beta}{c} \frac{\partial}{\partial t'} \right) + n_x \frac{\partial}{\partial x'} + n_y \frac{\partial}{\partial y'} + \left(\gamma n_z \frac{\partial}{\partial z'} - \beta \gamma \frac{\partial}{\partial z'} \right)$$

$$= \frac{\gamma}{c} (1 - n_z \beta) \frac{\partial}{\partial t'} + n_x \frac{\partial}{\partial x'} + n_y \frac{\partial}{\partial y'} + \gamma (n_z - \beta) \frac{\partial}{\partial z'}$$

From (89.9) $\gamma(1 - n_z \beta) = \gamma' / v$; $n_x = n'_x \gamma' / v$; $n_y = n'_y \gamma' / v$; $\gamma(n_z - \beta) = n'_z \gamma' / v$

hence

$$\frac{1}{c} \frac{\partial}{\partial t} + \underline{n} \cdot \nabla = \frac{1}{c} \frac{\gamma'}{v} \frac{\partial}{\partial t'} + \frac{\gamma'}{v} n'_x \frac{\partial}{\partial x'} + \frac{\gamma'}{v} n'_y \frac{\partial}{\partial y'} + \frac{\gamma'}{v} n'_z \frac{\partial}{\partial z'}$$

$$= \frac{\gamma'}{v} \left[\frac{1}{c} \frac{\partial}{\partial t'} + \underline{n}' \cdot \nabla' \right]$$

(92.13)

Now $I_\nu = (\nu^3/\nu'^3) I_{\nu'}$; $\eta_\nu = (\nu^2/\nu'^2) \eta_{\nu'}$; $\chi_\nu = (\nu'/\nu) \chi_{\nu'}$
 hence

$$(92.15) \quad \eta_{\nu'} \chi_{\nu'} I_\nu = (\nu^2/\nu'^2) [\eta_{\nu'} - \chi_{\nu'} I_{\nu'}]$$

and $\left\{ \frac{\nu'}{\nu} \left[\frac{1}{c} \frac{\partial}{\partial t'} + \underline{n}' \cdot \nabla' \right] \right\} \left\{ (\nu^3/\nu'^3) I_{\nu'} \right\} = (\nu^2/\nu'^2) [\eta_{\nu'} - \chi_{\nu'} I_{\nu'}]$;
 when relative motion is constant $\frac{\partial}{\partial x^k} [(\nu^3/\nu'^3) I_{\nu'}] = (\nu^3/\nu'^3) \frac{\partial I_{\nu'}}{\partial x^k}$
 and $\frac{\nu^2}{\nu'^2} \left\{ \left[\frac{1}{c} \frac{\partial}{\partial t'} + \underline{n}' \cdot \nabla' \right] I_{\nu'} \right\} = \frac{\nu^2}{\nu'^2} \left\{ \eta_{\nu'} - \chi_{\nu'} I_{\nu'} \right\}$
 which yields (92.11) by cancelling (ν^2/ν'^2) .

Section 7.2: The Dynamical Equations for a Radiating Fluid

Section 9.3: The Inertial-Frame Transfer Equation for a Moving Fluid

$$\chi(\nu, \underline{n}) = \frac{\nu_0}{\nu} \chi_0(\nu_0) = \gamma(1 - \beta \cdot \underline{n}) \chi_0[\gamma\nu(1 - \beta \cdot \underline{n})]$$

To order v/c , $\gamma \rightarrow 1$. Then $\chi_0(\nu - \nu \beta \cdot \underline{n}) \approx \chi_0(\nu) - \nu \beta \cdot \underline{n} \frac{\partial \chi_0(\nu)}{\partial \nu}$

and

$$(93.2) \quad \begin{aligned} \chi(\nu, \underline{n}) &\approx (1 - \beta \cdot \underline{n}) \left[\chi_0(\nu) - \nu \beta \cdot \underline{n} \frac{\partial \chi_0(\nu)}{\partial \nu} \right] \\ &\approx \chi_0(\nu) - \beta \cdot \underline{n} \chi_0(\nu) - \nu \beta \cdot \underline{n} \frac{\partial \chi_0(\nu)}{\partial \nu} \\ \chi(\nu, \underline{n}) &= \chi_0(\nu) - \beta \cdot \underline{n} \left[\chi_0(\nu) + \nu \frac{\partial \chi_0(\nu)}{\partial \nu} \right] \end{aligned}$$

$$\eta(\nu, \underline{n}) = \frac{\nu^2}{\nu_0^2} \eta_0(\nu_0) \approx [1 + \underline{n} \cdot \beta]^2 \left[\eta_0(\nu) - \nu \underline{n} \cdot \beta \frac{\partial \eta_0(\nu)}{\partial \nu} \right]$$

$$(93.3) \quad \begin{aligned} &\approx \eta_0(\nu) + 2 \underline{n} \cdot \beta \eta_0(\nu) - \nu \underline{n} \cdot \beta \frac{\partial \eta_0(\nu)}{\partial \nu} \\ \eta(\nu, \underline{n}) &= \eta_0(\nu) + \underline{n} \cdot \beta [2\eta_0(\nu) - \nu \frac{\partial \eta_0(\nu)}{\partial \nu}] \end{aligned}$$

Then

$$(93.4) \quad \begin{aligned} \left(\frac{1}{c} \frac{\partial}{\partial t} + \underline{n} \cdot \nabla \right) I(\nu, \underline{n}) &= \eta(\nu, \underline{n}) - \chi(\nu, \underline{n}) I(\nu, \underline{n}) \\ &= \eta_0 + \underline{n} \cdot \beta [2\eta_0(\nu) - \nu \frac{\partial \eta_0(\nu)}{\partial \nu}] - \left\{ \chi_0 - \beta \cdot \underline{n} (\chi_0 + \nu \frac{\partial \chi_0(\nu)}{\partial \nu}) \right\} I \\ \left(\frac{1}{c} \frac{\partial}{\partial t} + \underline{n} \cdot \nabla \right) I &= \eta_0 - \chi_0 I + \underline{n} \cdot \beta \left\{ 2\eta_0 - \nu \frac{\partial \eta_0(\nu)}{\partial \nu} + (\chi_0 + \nu \frac{\partial \chi_0(\nu)}{\partial \nu}) I \right\} \end{aligned}$$

moment equation:

$$\frac{1}{4\pi} \int d\omega \left\{ \frac{1}{c} \frac{\partial \mathbf{I}}{\partial t} + \underline{n} \cdot \nabla \mathbf{I} \right\} = \frac{1}{4\pi} \int d\omega \left\{ \eta_0 - \chi_0 \mathbf{I} + \beta \cdot \underline{n} \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \mathbf{I} \right] \right\}$$

$$= \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{1}{4\pi} \int \mathbf{I} d\omega \right] + \frac{1}{4\pi} \int \underline{n} \frac{\partial \mathbf{I}}{\partial x} d\omega = \eta_0 \frac{1}{4\pi} \int d\omega - \chi_0 \frac{1}{4\pi} \int \mathbf{I} d\omega$$

$$+ 2\beta \cdot \left[\frac{1}{4\pi} \int \underline{n} d\omega \right] - \nu \frac{\partial \eta_0}{\partial \nu} \beta \cdot \frac{1}{4\pi} \int \underline{n} d\omega + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta \cdot \frac{1}{4\pi} \int \underline{n} \mathbf{I} d\omega$$

Now $\int n^i d\omega = 0$ $\frac{1}{4\pi} \int \mathbf{I} d\omega = \mathbf{J}_v$ $\frac{1}{4\pi} \int n^i \mathbf{I} d\omega = H^i$, hence

$$\frac{1}{c} \frac{\partial \mathbf{J}_v}{\partial t} + \frac{1}{4\pi} \int n^i \frac{\partial \mathbf{I}}{\partial x^i} d\omega = \eta_0 - \chi_0 \mathbf{J}_v + \frac{2\beta_i}{4\pi} \int n^i d\omega - \nu \frac{\partial \eta_0}{\partial \nu} \frac{\beta_i}{4\pi} \int n^i d\omega$$

$$+ \chi_0 \beta_i H^i + \nu \frac{\partial \chi_0}{\partial \nu} \beta_i H^i$$

To evaluate $\frac{1}{4\pi} \int n^i \frac{\partial \mathbf{I}}{\partial x^i} d\omega$, we assume (?) $\frac{\partial n^i}{\partial x^i} = 0$, then it becomes

$$\frac{1}{4\pi} \frac{\partial}{\partial x^i} \int n^i \mathbf{I} d\omega = \frac{\partial H^i}{\partial x^i}$$

hence

$$(93.5) \quad \frac{1}{c} \frac{\partial \mathbf{J}_v}{\partial t} + \frac{\partial H^i}{\partial x^i} = \eta_0 - \chi_0 \mathbf{J}_v + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta_i H^i$$

First Moment Equation:

$$\frac{1}{4\pi} \int \underline{n} d\omega \frac{1}{c} \frac{\partial \mathbf{I}}{\partial t} + \frac{1}{4\pi} \int \underline{n} d\omega (\underline{n} \cdot \nabla \mathbf{I}) = \frac{\eta_0}{4\pi} \int \underline{n} d\omega - \frac{\chi_0}{4\pi} \int \underline{n} \mathbf{I} d\omega + 2\eta_0 \beta \cdot \frac{1}{4\pi} \int \underline{n} \underline{n} d\omega$$

$$- \nu \frac{\partial \eta_0}{\partial \nu} \frac{\beta}{4\pi} \cdot \int \underline{n} \underline{n} d\omega + \chi_0 \frac{\beta}{4\pi} \cdot \int \underline{n} \underline{n} \mathbf{I} d\omega + \nu \frac{\partial \chi_0}{\partial \nu} \frac{\beta}{4\pi} \int \underline{n} \underline{n} \mathbf{I} d\omega$$

or $\frac{1}{4\pi} \frac{1}{c} \frac{\partial}{\partial t} \int n^i \mathbf{I} d\omega + \frac{1}{4\pi} \int n^i \left(n^j \frac{\partial \mathbf{I}}{\partial x^j} \right) d\omega = \frac{\eta_0}{4\pi} \int n^i d\omega - \frac{\chi_0}{4\pi} \int n^i \mathbf{I} d\omega + \frac{2\eta_0}{4\pi} \beta_j \int n^i n^j d\omega$

$$- \nu \frac{\partial \eta_0}{\partial \nu} \frac{\beta_j}{4\pi} \int n^i n^j d\omega + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \frac{\beta_j}{4\pi} \int n^i n^j \mathbf{I} d\omega$$

$$\frac{1}{c} \frac{\partial H^i}{\partial t} + \frac{\partial K^i_j}{\partial x^j} = -\chi_0 H^i + \frac{2\eta_0}{3} \beta_i - \nu \frac{\partial \eta_0}{\partial \nu} \frac{\beta_i}{3} + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta_j K^i_j$$

$$(93.6) \quad \frac{1}{c} \frac{\partial H^i}{\partial t} + \frac{\partial K^i_j}{\partial x^j} = -\chi_0 H^i + \frac{\beta_i}{3} \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} \right] + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta_j K^i_j$$

In $\int n^i n^j d\omega$, the integral = 0 when $i \neq j$. Pick μ and ϕ so that

(93.7) $n^i = \mu$. By definition $dw = \mu d\phi$. Hence for $i=j$ the integral is $\int_0^{2\pi} d\phi \int_{-1}^1 \mu^2 d\mu = 2\pi \int_{-1}^1 \mu^2 d\mu = 2\pi \left[\frac{\mu^3}{3} \right]_{-1}^1 = 2\pi \cdot \frac{2}{3} = \frac{4\pi}{3}$

To write (93.5) and (93.6) in terms of E_ν , F_ν^i and P_ν^{ij} , use:

$$J_\nu = \frac{\sigma}{4\pi} E_\nu \quad H_\nu^i = \frac{1}{4\pi} F_\nu^i \quad K_\nu^{ij} = \frac{\sigma}{4\pi} P_\nu^{ij}$$

(93.5) becomes

$$\frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^i}{\partial x^i} = 4\pi\eta_0 - c\chi_0 E_\nu + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta_i F_\nu^i$$

or

$$(93.5a) \quad \frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^i}{\partial x^i} = 4\pi \left[\eta_0 - \chi_0 J_\nu \right] + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \beta_i F_\nu^i$$

and (93.6) becomes:

$$(93.5b) \quad \frac{1}{c} \frac{\partial F_\nu^i}{\partial t} + c \frac{\partial P_\nu^{ij}}{\partial x^j} = -\chi_0 F_\nu^i + \frac{4\pi\beta_i}{3} \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} \right] + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] \nu_j P_\nu^{ij}$$

$$\text{note: } 2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} = 3\eta_0 - \partial(\nu\eta_0)/\partial \nu$$

Integrating over frequency, (93.5a) becomes the energy equation:

$$(93.8) \quad \frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi \int (\eta_0 - \chi_0 J_\nu) d\nu + \beta_i \int \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] F_\nu^i d\nu = cG^0$$

and (93.6a) becomes the momentum equation:

$$(93.9) \quad \frac{1}{c^2} \frac{\partial F^i}{\partial t} + \frac{\partial P^{ij}}{\partial x^j} = -\frac{1}{c} \int \chi_0 F_\nu^i d\nu + \frac{4\pi\nu_j}{3c^2} \int \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} \right] d\nu + \frac{\nu_j}{c} \int \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu} \right] P_\nu^{ij} d\nu = -G^i$$

$$\text{where } \int \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} \right] d\nu = \int 3\eta_0 d\nu - \int \partial(\nu\eta_0)/\partial \nu d\nu = 3\int \eta_0 d\nu - 0$$

For grey material, χ_0 is not a fn. of ν and $\eta_0(\nu)/\chi_0(\nu) = B_\nu$, the Planck function

(93.8) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi\chi_0 \int (B_\nu - J_\nu) d\nu + \chi_0 \beta_i \int F_\nu^i d\nu ;$$

using κ for χ_0 and integrating:

$$\frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi\kappa (B - cE/4\pi) + \kappa \beta_i F^i$$

$$(93.10) \quad E_{,t} + F^i_{,i} = \kappa (4\pi B - cE) + \kappa \beta_i F^i$$

$$(93.9) \quad c^2 F_{,t}^i + P_{,j}^{ij} = -c^i \int \chi_0 F_{,j}^i dV + 4\pi v^i c^2 \int \eta_0 dV + v^j c^i \int [\chi_0 + v \frac{\partial \chi_0}{\partial v}] P_{,j}^{ij} dV$$

for grey material:

$$(93.11) \quad c^2 F_{,t}^i + P_{,j}^{ij} = -c^i \chi F^i + 4\pi v^i c^2 \chi B + v^j c^i \chi P_{,j}^{ij}$$

$$(91.16a): \quad E \approx E_0 + 2v c^2 F_0$$

if $E_0 \approx 4\pi B/c$ (diffusion regime) then

$$E = 4\pi B/c + 2v c^2 F_0$$

$$\text{or } 4\pi B - cE = -2\beta^c E_0$$

$$\text{and } E_{,t} + F_{,i}^i = \alpha(4\pi B - cE) + \alpha\beta^c E \\ = \alpha\beta^c (E - 2E_0)$$

$$\text{From (91.17a), } F = F_0 + v(E_0 + P_0)$$

thus:

$$E_{,t} + F_{,i}^i = \alpha\beta^c [E_0 + v(E_0 + P_0) - 2E_0] \\ = \alpha\beta^c [-E_0 + v(E_0 + P_0)]$$

$$(93.12) \quad E_{,t} + F_{,i}^i \approx \alpha\beta^c (-E_0) \text{ in the static diffusion regime.}$$

Order of magnitude estimate of $(\alpha v \cdot E / c) / \nabla \cdot E$:

$$\alpha \sim 1/\lambda_p; \quad \nabla \cdot E \sim E/l, \text{ hence}$$

$$(\alpha v \cdot E / c) / \nabla \cdot E \approx (vF/c\lambda_p)(F/l) \sim lv/c\lambda_p$$

Order of magnitude estimate of $(c^2 F_{,t}^i) / (\alpha F^i / c)$

$$\approx (c^2 F / t_f) / (\lambda_p^{-1} F / c)$$

and $l/t_f \sim v$, hence $t_f^{-1} \sim vl^{-1}$ and

$$\text{ratio} \approx (v/c)(\lambda_p) = v\lambda_p/c$$

Thus (93.11) becomes

$$P_{,j}^{ij} \approx -\alpha c^i F^i + \alpha v^i c^i (4\pi B/c) + \alpha c^i v_j P_{,j}^{ij}$$

$$\alpha c^i F^i = \alpha v^i c^2 4\pi B + \alpha c^i v_j P_{,j}^{ij} - P_{,j}^{ij}$$

$$(93.13) \quad F^i = 4\pi v^i c^i B - (c/\alpha) P_{,j}^{ij} + v_j P_{,j}^{ij}$$

$$\eta - \chi I \rightarrow \eta - \chi S = \chi \left(\frac{\eta}{\chi} - S \right) \quad \text{Diffusion limit } \eta/\chi = B, \text{ hence}$$

$$\eta - \chi S = \chi(B - S) = (\chi/4\pi)(4\pi B - cE) \quad \text{as } S = cE/4\pi$$

Assume (show in §97) that diffusion limit gives: $E_0 \approx 4\pi B/c$;
 $P_0^{ij} = P_0^{ij} + O(\lambda_p v/c) = \frac{1}{3} E_0 \delta^{ij} + O(\lambda_p v/c)$; $F_0^i = -c \chi^{-1} \nabla \cdot (P_0^i)$

then

$$F^i = v^i E_0 + v_j [P_0^{ij} + O(\lambda_p v/c)] - (c/\chi) (P_0^{ij} + O(\lambda_p v/c))_{,j}$$

$$(93.14) \quad F^i \approx v^i E_0 + v_j P_0^{ij} - F_0^i \approx F_0^i + \frac{4}{3} E_0 v^i$$

Streaming limit: $l/\lambda_p \leq 1$; $E \approx P$; $F \approx cE$

Diffusion limit: $E \approx 3P$

Static diffusion: $t_f \gg t_d$ ($v/c \ll (\lambda_p/l)$) ; $(F/cE) = O(\lambda_p/l)$

Dynamic diffusion: $t_f \lesssim t_d$ ($v/c \gtrsim (\lambda_p/l)$) ; $F \sim (cE)(v/c) \approx (vE)$

Equation (91.17a): $F \approx F_0 + v(E_0 + P_0)$

Dynamic diffusion $v(E_0 + P_0) \gg F_0$; $E_0 \approx 3P_0$

$$F \approx v \cdot \frac{4}{3} E_0 \quad \text{and } (F/cE) \sim O(v/c)$$

For estimate of $S = \chi [4\pi B - cE]$ 80.20:

$$c^{-1} \partial E / \partial t = \frac{1}{3} \nabla \cdot (\chi^{-1} \nabla E) + (4\pi/c) \eta - \chi E$$

$$\text{or } c^{-1} \partial E / \partial t = \frac{1}{3} \nabla \cdot (\chi^{-1} \nabla E) + c^{-1} \chi [4\pi B - cE]$$

$$\text{Scaling: } c^{-1} E / t_f \quad \frac{1}{3} \frac{1}{l} \lambda_p \frac{E}{l} \quad - c^{-1} S$$

$$c^{-1} E v / l \quad \frac{1}{3} E \lambda_p / l^2 \quad c^{-1} S$$

for dynamic diffusion, where time derivative dominates over ∇^2 term

$$c^{-1} S \sim c^{-1} E v / l \Rightarrow S \sim E (v/l)$$

for static diffusion where $\partial/\partial t$ term is ignored:

$$c^{-1} S \sim \frac{1}{3} E \lambda_p / l^2 \Rightarrow S \sim E (\lambda_p c / l^2)$$

Streaming limit

Transfer equation (93.4): on a fluid-flow timescale $t_f = l/v$

$$(a) \quad c^{-1} \frac{\partial I}{\partial t} + \underline{u} \cdot \nabla I = \eta_0 - \chi_0 I + \left(\frac{\eta v}{c} \right) \left\{ 2\eta_0 - v \frac{\partial \eta_0}{\partial v} + [\chi_0 + v \frac{\partial \chi_0}{\partial v}] I \right\}$$

$$\frac{I}{c t_f} \quad \frac{I}{l} \quad \frac{I}{\lambda_p} \quad \frac{I}{\lambda_p} \quad \frac{v}{c} \frac{I}{\lambda_p}$$

multiply them by l/I

(a) $\frac{l}{ct_f} = \frac{lv}{cl} = \frac{v}{c} \left(1 ; \frac{l}{\lambda_p} ; \frac{l}{\lambda_p} ; \frac{v}{c} \frac{l}{\lambda_p} \right)$

on a radiation-flow timescale t_R ,

(b) $\frac{1}{c} \frac{\partial I}{\partial t} = \frac{1}{c} \frac{1}{t_R} = \frac{1}{c} \frac{c}{l} = \frac{1}{l}$ and $\frac{l}{I} \cdot \left(\frac{l}{I} \right) = 1$

Transfer equation: diffusion limit

$$c^{-1} \frac{\partial I}{\partial t} + \underline{n} \cdot \nabla I = \chi_0 [n_0/\chi_0 - I] + \underline{n} \cdot \underline{\beta} \left[2n_0 - v \frac{\partial n_0}{\partial \omega} + (\chi_0 + v \frac{\partial \chi_0}{\partial \omega}) I \right]$$

$$\sim \lambda_0 \left[n_0/\chi_0 - \frac{c}{4\pi} E \right]$$

$$\left(\frac{I}{ct_f} \right) \left(\frac{I}{l} \right) \quad \frac{1}{4\pi\lambda_p} \left[\sim 4\pi B - cE \right] \quad \frac{v}{c} \frac{l}{\lambda_p} \left[0(4+B-cE) \right]$$

Static diffusion:

$$\left(\frac{I}{l} \right) \left(\frac{v}{c} \right) \quad \left(\frac{I}{l} \right) \quad E \left(\frac{c\lambda_p}{l} \right) \quad \left(\frac{v}{c} \right) \left(\frac{E\lambda_p}{l} \right) \left(\frac{c}{4\pi} \right) \left(\frac{l}{\lambda_p} \right)$$

$$I/l \left\{ \frac{v}{c} \quad 1 \quad \frac{E}{I} \left(\frac{c\lambda_p}{l} \right) \quad \left(\frac{v\lambda_p}{c} \right) \left(\frac{E}{I} \right) \left(\frac{l}{\lambda_p} \right) \right.$$

But $E/I \sim 4\pi/c$ hence term order is.

(c) $\frac{v}{c} \quad 1 \quad (\lambda_p/l) \quad \left(\frac{v}{c} \right) \left(\frac{\lambda_p}{l} \right) + \left(\frac{v}{c} \right) \left(\frac{l}{\lambda} \right)$

Dynamic diffusion:

$$\left(\frac{I}{l} \right) \left(\frac{v}{c} \right) \quad \left(\frac{I}{l} \right) \quad E \left(\frac{v}{l} \right) \quad E \left(\frac{v}{c} \right) \left(\frac{v}{l} \right); \left(\frac{v}{c} \right) I/\lambda_p$$

$$E \sim (4\pi/c) I \Rightarrow$$

$$\left(\frac{I}{l} \right) \left(\frac{v}{c} \right) \quad \left(\frac{I}{l} \right) \quad \left(\frac{I}{c} \right) \left(\frac{v}{c} \right) \quad \left(\frac{I}{c} \right) \left(\frac{v}{c} \right) \left(\frac{v}{c} \right); \left(\frac{v}{c} \right) \left(\frac{I}{l} \right) \left(\frac{l}{\lambda_p} \right)$$

(d) or $\frac{v}{c} \quad 1 \quad \frac{v}{c} \quad \frac{v^2}{c^2}; \left(\frac{v}{c} \right) \left(\frac{l}{\lambda_p} \right)$

Radiative momentum equation (93.)

$$c^2 F_{,t}^i + P^{ij}_{,j} = -c^{-1} \int \chi_0 F^i dv + 4\pi c^2 v^i \int n_0 dv + c^{-1} v_j \int (\chi_0 + v \partial \chi_0 / \partial \nu) P^{ij} dv$$

(a) Streaming limit; fluid-flow timescale

$$c^2 F/t_f \quad P/l \quad c^{-1} F/\lambda_p \quad c^{-2} v \lambda_p^{-1} I \quad c^{-1} v \lambda_p^{-1} P$$

now $F \approx cE$, $E \approx P$, $t_f^{-1} \sim v/l$

$$c^2 (cE) v/l \quad E/l \quad c^{-1} (cE)/\lambda_p \quad c^{-2} v \lambda_p^{-1} (cE) \quad c^{-1} v \lambda_p^{-1} E$$

(a) $(E/e) \left\{ \begin{matrix} v/c & 1 & l/\lambda_p & (v/c)(l/\lambda_p) & (v/c)(l/\lambda_p) \end{matrix} \right\}$

(b) Radiation time scale (streaming) $c^{-1} \partial/\partial t \rightarrow c^{-1} t_e^{-1} \approx l^{-1}$ and time-der term is $O(1)$

$$\left\{ \begin{matrix} 1 & 1 & l/\lambda_p & (v/c)(l/\lambda_p) & (v/c)(l/\lambda_p) \end{matrix} \right\}$$

(c) Static diffusion:

$$\begin{matrix} c^{-2} F/t_f & P/l & c^{-1} F/\lambda_p & c^{-2} v I/\lambda_p & c^{-1} v \lambda_p^{-1} P \\ c^{-2} F(v/e) & E/e & c^{-1} F/\lambda_p & c^{-2} v \lambda_p^{-1} (cE) & c^{-1} v \lambda_p^{-1} E \\ F \sim cE \lambda_p/l & & & & \\ c^2 (cE \lambda_p v/c^2) & E/e & c^{-1} (cE \lambda_p/e \lambda_p) & c^{-1} v \lambda_p^{-1} E & c^{-1} v \lambda_p^{-1} E \\ E(c^{-1} \lambda_p v/c^2) & E/e & (E/e) & (v/c) E/\lambda_p & (v/c) (E/\lambda_p) \\ (E/e) \left\{ \begin{matrix} (v/c)(\lambda_p/l) & 1 & 1 & (v/c)(l/\lambda_p) & (v/c)(l/\lambda_p) \end{matrix} \right\} \end{matrix}$$

(d) Dynamic diffusion: $F \sim cE(v/c)$

$$\begin{matrix} c^{-2} (cE v/c)(v/e) & (E/e) & c^{-1} \lambda_p^{-1} (vE) & c^{-2} v c E/\lambda_p & c^{-1} v \lambda_p^{-1} E \\ (E/e)(v^2/c^2) & (E/e) & (E/e)(v/c)(l/\lambda_p) & (E/e)(v/c)(l/\lambda_p) & (E/e)(v/c)(l/\lambda_p) \\ v^2/c^2 & 1 & (v/c)(l/\lambda_p) & (v/c)(l/\lambda_p) & (v/c)(l/\lambda_p) \end{matrix}$$

Radiative Energy Equation (93.8)

$$E_{,t} + F_{,x}^i = 4\pi \int \eta_0 d\nu - 4\pi \int \chi_0 J d\nu + (v/c) \int [\chi_0 + v \partial \chi_0 / \partial \nu] F d\nu$$

(a) $E/t_f \quad F/l \quad J/\lambda_p \quad J/\lambda_p \quad (v/c) F/\lambda_p$

in streaming limit $cE \approx F$; also $J \sim cE$

Fluid-flow timescale

$$\begin{matrix} E(v/c) & cE/l & cE/\lambda_p & cE/\lambda_p & (v/c)(cE/\lambda_p) \\ cE/e \left\{ \begin{matrix} v/c & 1 & l/\lambda_p & l/\lambda_p & (v/c)(l/\lambda_p) \end{matrix} \right\} \\ \text{on radiation timescale, streaming limit:} \\ 1 & 1 & l/\lambda_p & l/\lambda_p & (v/c)(l/\lambda_p) \end{matrix}$$

(b) Static Diffusion

$$E_{,t} + F^i_{,i} = 4\pi \int (\eta_0 - \lambda_0 J) d\nu + (v/c) \int [\lambda_0 + v \partial \lambda_0 / \partial \nu] F d\nu$$

E/t_f	F/e	$E(\lambda_p c / l^2)$	$(v/c) \lambda_p^{-1} F$
$E v / e$	$(c E \lambda_p / l^2)$	$E(\lambda_p c / l^2)$	$(v/c) \lambda_p^{-1} (c E \lambda_p / l^2)$
$(c E / e) (\lambda_p / l^2)$	$(v/c) (c / \lambda_p)$	$\underline{1}$	$(v/c) (l / \lambda_p)$

(c) Dynamic Diffusion

$E v / e$	$l^{-1} (E v)$	$E (v / e)$	$(v/c) \lambda_p^{-1} (v E)$
$(E v / e) \{$	$\underline{1}$	$\underline{1}$	$(v/c) l / \lambda_p \}$

§94 Inertial-Frame Equations of Radiation Hydrodynamics

(91.8) $R = (\mathcal{R}) = \begin{pmatrix} E & c^1 F & 0 & 0 \\ c^1 F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E-P) \end{pmatrix}$ physical components
 rewritten with r the 1st spatial coord. (t, r, θ, ϕ)

(91.9) $R^{\alpha\beta} = \begin{pmatrix} E & c^1 F & 0 & 0 \\ c^1 F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P)/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E-P)/r^2 \sin^2 \theta \end{pmatrix}$ contravariant components
 in (t, r, θ, ϕ)
 rewritten in E, F, P

Use (A3.91) on (91.8) with $\partial/\partial\theta = \partial/\partial\phi = 0$

$$\begin{aligned} R^{1j}_{;j} &= r^{-2} \partial(r^2 R_{11})/\partial r - r^{-1} (R_{22} + R_{33}) \\ &= r^{-2} \partial(r^2 P)/\partial r - r^{-1} [\frac{1}{2}(E-P) + \frac{1}{2}(E-P)] \\ &= \partial P/\partial r + 2rP - r^{-1}(E-P) \end{aligned}$$

$$R^{1j}_{;j} = \partial P/\partial r + r^{-1} [2P - E + P] = \partial P/\partial r + r^{-1} [3P - E]$$

$$\begin{aligned} -G^1 &= R^{10}_{;0} + R^{1\beta}_{;\beta} \\ &= c^1 \frac{\partial}{\partial t} (c^1 F) + \partial P/\partial r + [3P - E]/r \end{aligned}$$

(94.5) $-G^1 = c^2 \partial F/\partial t + \partial P/\partial r + (3P - E)/r$

$$\begin{aligned} -G^0 &= R^{00}_{;t} + R^{0j}_{;j} = c^1 \frac{\partial}{\partial t} (E) + R^{01}_{;1} \\ &= c^1 \partial E/\partial t + r^{-2} \partial(r^2 c^1 F)/\partial r \end{aligned}$$

(94.4) or $-cG^0 = \partial E/\partial t + r^{-2} \partial(r^2 F)/\partial r$

To write (94.6) $M^{\alpha\beta}_{;\beta} = F^\alpha + G^\alpha$ and (94.8) $(M^{\alpha\beta} + R^{\alpha A})_{;\beta} = F^\alpha$ in cartesian coordinates, we need $M^{\alpha\beta}_{;\beta}$ from §42

$$(\rho_i - P/c^2)_{;t} + (\rho_i v^j)_{;j} = v_j f^j/c^2 \quad (42.3)$$

$$(\rho_i v^i)_{;t} + (\rho_i v^i v^j + p \delta^{ij})_{;j} = f^i \quad (42.4)$$

Then (94.6) becomes:

$$(94.9a) \quad \begin{aligned} \bar{c}^i m^{0\beta}_{;\beta} &= (\rho_i - p/c^2)_{,t} + (\rho_i v^j)_{,j} = v_j f^j / c^2 + c^{-1} G^0 \\ \Rightarrow (\rho_i c^2 - p)_{,t} + (\rho_i c^2 v^j)_{,j} &= v_j f^j + c G^0 \end{aligned}$$

and

$$(94.10a) \quad \begin{aligned} m^{i\beta}_{;\beta} &= (m^{i0})_{,0} + (m^{ij})_{,j} \\ &= c^i (c p v^i)_{,t} + (p v^i v^j + p \delta^{ij})_{,j} = f^i + G^i \\ m^{i\beta}_{;\beta} &= (p v^i)_{,t} + (p v^i v^j + p \delta^{ij})_{,j} = f^i + G^i \\ \text{or } (p v^i)_{,t} + (p v^i v^j)_{,j} &= f^i - p_{,i} + G^i \end{aligned}$$

And (94.8) becomes:

$$(94.9a) \quad \begin{aligned} c m^{0\beta}_{;\beta} + c R^{0\beta}_{;\beta} &= (\rho_i c^2 - p)_{,t} + (c^2 \rho_i v^j)_{,j} + E_{,t} + F^i_{,j} \\ \Rightarrow c(m^{0\beta} + R^{0\beta})_{;\beta} &= (\rho_i c^2 - p + E)_{,t} + (c^2 \rho_i v^j + F^j)_{,j} = v_j f^j \end{aligned}$$

and

$$(94.10b) \quad \begin{aligned} (m^{i\beta} + R^{i\beta})_{;\beta} &= c^i (p_i c v^i)_{,t} + (p_i v^i v^j + p \delta^{ij})_{,j} + c^i (c^{-1} F^i)_{,t} + P^i_{,j} \\ &= (p_i c^i)_{,t} + (p_i v^i v^j + p \delta^{ij})_{,j} + c^2 F^i_{,t} + P^i_{,j} = f^i \\ \text{or} \\ (m^{i\beta} + R^{i\beta})_{;\beta} &= (p_i v^i + c^2 F^i)_{,t} + (p_i v^i v^j + P^i_{,j})_{,j} = f^i - p_{,i} \end{aligned}$$

Subtract $c^2 \times$ continuity equation $(\delta p_0)_{,t} + (\delta p v^j)_{,j} = 0$ from 94.9a and 94.9b:

First write both in terms of p and e : $\rho_i = \gamma^2 \rho_0 (1 + h/c^2)$.

$$(94.9a) \text{ becomes: } \begin{aligned} \rho_i &\equiv \gamma^2 \rho_{000} \equiv \gamma^2 \rho_0 (1 + h/c^2) \\ \delta \rho_0 &\equiv p \\ h &= e + \delta p / \rho \end{aligned} \quad \begin{aligned} &= \delta p (1 + e/c^2 + \delta p / \rho c^2) \\ &[\delta p c^2 + \delta p h - p]_{,t} + [\delta p c^2 v^j + \delta p h v^j]_{,j} = v_j f^j + c G^0 \\ \Rightarrow &[\delta p c^2 + \delta p (e + \delta p / \rho) - p]_{,t} + [\delta p c^2 v^j + \delta p (e + \delta p / \rho) v^j]_{,j} = v_j f^j + c G^0 \\ \Rightarrow &[\delta p c^2 + \delta p e + (\delta^2 - 1) p]_{,t} + [\delta p c^2 v^j + \delta p e v^j + \delta^2 p v^j]_{,j} = v_j f^j + c G^0 \\ &\quad - (c^2 p)_{,t} \quad - (c^2 p v^j)_{,j} = 0 \end{aligned}$$

$$(94.11a) \quad [(\delta - 1) p c^2 + \delta p e + (\delta^2 - 1) p]_{,t} + \{[(\delta - 1) p c^2 + \delta p e + \delta^2 p] v^j\}_{,j} = v_j f^j + c G^0$$

(94.9b) becomes:

$$\frac{[\gamma \rho c^2 (1 + v/c^2 + \gamma p/\rho c^2) - p + E]_{,t} + [\gamma \rho c^2 v^j (1 + v/c^2 + \gamma p/\rho c^2) + F^j]_{,j} - (c^2 \rho)_{,t} - (c^2 \rho v^j)_{,j}}{=} = v_j f^j = 0$$

(94.11b) $[(\gamma-1)\rho c^2 + \gamma p + (\gamma^2-1)p]_{,t} + \{(\gamma-1)\rho c^2 + \gamma p + \gamma^2 p\} v^j_{,j} + F^j_{,j} = v_j f^j$

Momentum Eqs (94.10)

$$\begin{aligned} (\gamma^2 \rho_{000} v^i)_{,t} + (\gamma^2 \rho_{000} v^i v^j)_{,j} &= f_i - p_{,i} + G_i \\ v^i/c^2 (\gamma^2 \rho_{000} c^2 - p)_{,t} + v^i/c^2 (\gamma^2 \rho_{000} c^2 v^j)_{,j} &= (v^i/c^2) v_j f^j + (v^i/c^2) e G^0 \end{aligned}$$

$$\begin{aligned} &[\gamma^2 \rho_{000} v^i]_{,t} - v^i (\gamma^2 \rho_{000} - p/c^2)_{,t} + (\gamma^2 \rho_{000} v^i v^j)_{,j} - v^i (\gamma^2 \rho_{000} v^j)_{,j} \\ &= f_i - (v^i v_j / c^2) f^j - p_{,i} + G_i - (v^i/c) G^0 \\ = &v^i (\cancel{\gamma^2 \rho_{000}})_{,t} + \gamma^2 \rho_{000} v^i_{,t} + v^i (p/c^2)_{,t} + v^i (\cancel{\gamma^2 \rho_{000} v^j})_{,j} + \gamma^2 \rho_{000} v^j v^i_{,j} - v^i (\cancel{\gamma^2 \rho_{000} v^j})_{,j} \\ &- v^i (\cancel{\gamma^2 \rho_{000}})_{,t} = f_i - f_j v^i v^j / c^2 - p_{,i} + G_i - (v^i/c) G^0 \end{aligned}$$

=>

$$\begin{aligned} \gamma^2 \rho_{000} v^i_{,t} + (v^i/c^2) p_{,t} + \gamma^2 \rho_{000} v^j v^i_{,j} &= f_i - f_j v^i v^j / c^2 - p_{,i} + G_i - (v^i/c) G^0 \\ \gamma^2 \rho_{000} [v^i_{,t} + v^j v^i_{,j}] &= f_i - f_j v^i v^j / c^2 - p_{,i} - (v^i/c^2) p_{,t} + G_i - (v^i/c) G^0 \end{aligned}$$

(94.12a) $\Rightarrow \rho_i [Dv^i/Dt] = \underline{f} - \underline{\nabla} p - (\underline{v}/c^2) (\underline{v} \cdot \underline{f} + \partial p/\partial t) + \underline{G} - \underline{v} c^i G^0$

$$\begin{aligned} (p_i v^i + c^2 F^i)_{,t} + (p_i v^i v^j + P^{ij})_{,j} &= f^i - p_{,i} \\ (v^i/c^2) (p_i c^2 - p + E)_{,t} + (v^i/c^2) (p_i c^2 v^j + F^i)_{,j} &= v_j f^j (v^i/c^2) \end{aligned}$$

$$\begin{aligned} p_i v^i_{,t} + \cancel{v^i p_{,i}} + c^2 F^i_{,t} - \cancel{v^i p_{,i}}_{,t} + (p v^i/c^2 - E v^i/c^2)_{,t} \\ + p_i v^j v^i_{,j} + v^i (\cancel{p_{,i}})_{,j} - v^i (\cancel{p_{,i}})_{,j} + p_{i,j} c^2 v^i F^j_{,j} &= f^i - p_{,i} - v^i v_j f^j / c^2 \\ p_i v^i_{,t} + p_i v^j v^i_{,j} + \frac{F^i_{,t} + p v^i c^{-2} - E v^i c^{-2}}{,t} + (p_{i,j} c^2 v^i F^j)_{,j} &= f^i - p_{,i} - v^i v_j f^j / c^2 \end{aligned}$$

(94.12b) $\rho_i (Dv^i/Dt) + c^2 [\underline{L} \cdot \underline{E}]_{,t} + (\underline{P}' - \underline{E}) \underline{v}] + [\underline{\nabla} \cdot \underline{P} - \underline{v} (\underline{\nabla} \cdot \underline{E}) c^2] = \underline{f} - \underline{\nabla} p - \underline{v} (\underline{v} \cdot \underline{f})$

$$\text{or } \rho, (D_{\underline{v}}/Dt) = \underline{f} - \underline{\nabla}P - \underline{v}c^2 [P, t + \underline{v} \cdot \underline{f}] - (\underline{\nabla} \cdot \underline{P} + c^2 \underline{E}, t) + \underline{v}c^2 (\underline{E}, t + \underline{\nabla} \cdot \underline{E})$$

Writing $dt = \gamma d\tau$ and $\rho_x = \gamma^{-1} \rho$, or $\rho = \gamma \rho_x$, we have

$$(94.12a) \quad \rho_x (D_{\underline{v}}/D\tau) = \underline{f} - \underline{\nabla}P - c^2 \underline{v} (P, t + \underline{f} \cdot \underline{v}) + \underline{G} - c^2 \underline{v} G^0$$

and

$$(94.12b) \quad \rho_x (D_{\underline{v}}/D\tau) = \underline{f} - \underline{\nabla}P - \underline{v}c^2 [\partial P/\partial t + \underline{v} \cdot \underline{f}] - (\underline{\nabla} \cdot \underline{P} + c^2 \partial \underline{E}/\partial t + \underline{v}c^2 (\partial \underline{E}/\partial t + \underline{\nabla} \cdot \underline{E}))$$

Dropping terms of higher order than (v/c) :

$$(94.13a) \quad \rho (D_{\underline{v}}/Dt) = \underline{f} - \underline{\nabla}P + \underline{G} - \underline{v}c^2 G^0$$

$$(94.13b) \quad \rho (D_{\underline{v}}/Dt) = \underline{f} - \underline{\nabla}P - (\underline{\nabla} \cdot \underline{P} + c^2 \partial \underline{E}/\partial t) + \underline{v}c^2 (\partial \underline{E}/\partial t + \underline{\nabla} \cdot \underline{E})$$

The Total Energy Equation

Take (94.11) and let $\gamma \rightarrow 1$, $(\gamma - 1) \rightarrow \frac{1}{2}v^2/c^2$, then

$$(94.15a) \quad [p v^2/2 + pe + p v^2/c^2], t + \{ [p v^2/2 + pe + p] v^i \}, i = v_i f^i + c G^0$$

$$(94.15b) \quad [p v^2/2 + pe + E], t + \{ [p v^2/2 + pe + p] v^i + F^i \}, i = v_i f^i$$

$$\begin{aligned} [p(e + v^2/2)], t + \{ [p(e + v^2/2) + p] v^i \}, t &= (e + v^2/2) \partial p / \partial t + p (e + v^2/2), t \\ &+ (p v^i), i + (e + v^2/2), i p v^i + (e + v^2/2) (p v^i), i \\ &= (e + v^2/2) [\partial p / \partial t + \underline{\nabla} \cdot (\underline{p} \underline{v})] + p \partial (e + v^2/2) / \partial t + p \underline{v} \cdot \underline{\nabla} (e + v^2/2) + (p v^i), i \\ &\quad \text{"0 by continuity eqn."} \end{aligned}$$

$$(94.16a) \text{ hence } \rho D(e + v^2/2) / Dt + \underline{\nabla} \cdot (p \underline{v}) = \underline{v} \cdot \underline{f} + c G^0$$

Similarly,

$$[\rho(c + v^2/2) + E]_{,t} + \{[\rho(c + v^2/2) + p]v^i + F^i\}_{,i} = v_i f^i$$

becomes

$$(94.16b) \quad \rho D(c + v^2/2)/Dt + \nabla \cdot (p\underline{v}) + \partial E/\partial t + \nabla \cdot \underline{F} = \underline{v} \cdot \underline{f}$$

Mechanical Energy Equation

$$\underline{v} \cdot \{ \rho(D\underline{v}/Dt) = \underline{f} - \nabla p + \underline{G} - \underline{v}c'G^0 \}$$

$$\rho \underline{v} \cdot \partial \underline{v} / \partial t + \rho \underline{v} \cdot (\underline{v} \cdot \nabla) \underline{v} = \underline{v} \cdot \underline{f} - \underline{v} \cdot \nabla p + \underline{v} \cdot \underline{G} - v^2 c' G^0$$

$$\rho D(v^2/2)/Dt = -\underline{v} \cdot \nabla p + \underline{v} \cdot (\underline{f} + \underline{G})$$

$$\underline{v} \cdot \rho D\underline{v}/Dt = \underline{v} \cdot \underline{f} - \underline{v} \cdot \nabla p - c^2 \underline{v} \cdot \partial E / \partial t - \underline{v} \cdot \nabla \cdot \underline{P}$$

$$+ c^2 v^2 (\partial E / \partial t + \nabla \cdot \underline{F})$$

$$\Rightarrow \rho D(v^2/2)/Dt = -\underline{v} \cdot \nabla p - \underline{v} \cdot \nabla \cdot \underline{P} + \underline{v} \cdot \underline{f} - (v^2/c^2) \cdot \partial E / \partial t$$

$$= -\underline{v} \cdot \nabla p + \underline{v} \cdot \underline{f} - \underline{v} \cdot (c^2 \partial E / \partial t + \nabla \cdot \underline{P})$$

Gas Energy Equation

Start with: (94.6) $m^{\alpha\beta}_{, \beta} = F^\alpha + G^\alpha$

$$(94.9a) \quad (\rho_0 c^2 p)_{,t} + (\rho_0 c^2 v^j)_{,j} = v_i f^i + c G^0 = c m^{0\beta}_{, \beta}$$

$$(94.10a) \quad (\rho_0 v_i)_{,t} + (\rho_0 v_i v^j)_{,j} = f_i - p_{,i} + G_i = m^{i\beta}_{, \beta}$$

$$\rho_1 = \gamma p + \gamma^2 p c^2 + \gamma^2 p c^2$$

$$\text{Thus } m^{0\beta}_{, \beta} = c^{-1} [\gamma p c^2 + \gamma p c + (\gamma^2 - 1) p]_{,t} + [(\gamma p c^2 v^j + \gamma p c v^j + \gamma^2 p v^j)]_{,j}$$

$$\text{and } \gamma m^{i\beta}_{, \beta} = -\gamma [\gamma p c^2 + \gamma p c + (\gamma^2 - 1) p]_{,t} + [\gamma p c^2 v^j + \gamma p c v^j + \gamma^2 p v^j]_{,j}$$

add $\gamma c^2 \times$ [cont. eqn].

$$+ \gamma (p c^2)_{,t} + \gamma (p c^2 v^j)_{,j}$$

$$\Rightarrow -\gamma \{ (\gamma - 1) p c^2 + \gamma p c + (\gamma^2 - 1) p \}_{,t} - \gamma \{ (\gamma - 1) p c^2 v^j + \gamma p c v^j + \gamma^2 p v^j \}_{,j}$$

Take limit $\gamma \rightarrow 1$, $\gamma - 1 \rightarrow \frac{1}{2} \frac{v^2}{c^2}$, $\gamma^2 - 1 \rightarrow \frac{v^2}{c^2}$, then

$$\gamma_0 m^{0\beta}_{, \beta} \rightarrow - \{ \frac{1}{2} \rho v^2 + p c + \frac{v^2}{c^2} p \}_{,t} - \{ \frac{1}{2} \rho v^2 v^j + p c v^j + p v^j \}_{,j}$$

(and dropping $O(v^2/c^2)$ terms)

$$\begin{aligned} V_0 m^{0\beta}_{;\beta} &= -\left(\frac{1}{2}\rho v^2\right)_{,t} - (\rho e)_{,t} - \left(\frac{1}{2}\rho v^2 v^j\right)_{,j} - (\rho e v^j)_{,j} - (\rho v^j)_{,j} \\ &= -\left(\frac{1}{2}\rho v^2\right)_{,t} - \rho(De/Dt) - \left(\frac{1}{2}\rho v^2 v^j\right)_{,j} - (\rho v^j)_{,j} \end{aligned}$$

we also have

$$\rho_1 = \rho + \rho v^2/c^2 + \rho p/c^2 \quad V_i m^{i\beta}_{;\beta} = v_i (\rho_1 v_i)_{,t} + v_i (\rho_1 v_i v^j)_{,j} + v_i P_{,i}$$

$$= v_i \left[(\rho + \rho v^2/c^2 + \rho p/c^2) v_i \right]_{,t} + v_i \left[(\rho + \rho v^2/c^2 + \rho p/c^2) v_i v^j \right]_{,j} + v_i P_{,i}$$

letting $\delta \rightarrow 1$ and dropping $O(v^2/c^2)$ terms, we have:

$$\begin{aligned} V_i m^{i\beta}_{;\beta} &\cong v_i (\rho v_i)_{,t} + v_i (\rho v_i v^j)_{,j} + v_i P_{,i} \\ &= \left(\frac{1}{2}\rho v^2\right)_{,t} + \left(\frac{1}{2}\rho v^2 v^j\right)_{,j} + v_i P_{,i} \end{aligned}$$

Adding $V_0 m^{0\beta}_{;\beta} + V_i m^{i\beta}_{;\beta}$

$$\begin{aligned} &= -\left(\frac{1}{2}\rho v^2\right)_{,t} - \rho(De/Dt) - \left(\frac{1}{2}\rho v^2 v^j\right)_{,j} - (\rho v^j)_{,j} \\ &\quad + \left(\frac{1}{2}\rho v^2\right)_{,t} + \left(\frac{1}{2}\rho v^2 v^j\right)_{,j} + v_i P_{,i} \end{aligned}$$

$$\begin{aligned} &= -\rho(De/Dt) - (\rho v^j)_{,j} + v_i P_{,i} \\ &= -\rho(De/Dt) - v^i P_{,i} - \rho v^i_{,i} + v_i P_{,i} \\ &= -\rho(De/Dt) - \rho v^i_{,i} \end{aligned}$$

But $D\rho/Dt + \rho v^i_{,i} = 0 \Rightarrow v^i_{,i} = -\rho^{-1}(D\rho/Dt) = \rho(D\rho^{-1}/Dt)$

hence

$$V_0 m^{0\beta}_{;\beta} + V_i m^{i\beta}_{;\beta} = -\rho(De/Dt) - \rho \rho(D\rho^{-1}/Dt) = -V_2 F^\alpha + V_2 G^\alpha$$

or, finally

$$\rho(De/Dt) + \rho \rho(D\rho^{-1}/Dt) = -V_2 F^\alpha - V_2 G^\alpha$$

$$\text{or } \rho \left[De/Dt + \rho D(\rho^{-1})/Dt \right] = -V_2 F^\alpha - V_2 G^\alpha$$

$$(\rho_{000} V^\alpha V^\beta + p g^{\alpha\beta})_{;\beta} = F^\alpha + G^\alpha$$

$$\begin{aligned} V_\alpha (\rho_{000} V^\alpha V^\beta + p g^{\alpha\beta})_{;\beta} &= V_\alpha F^\alpha + V_\alpha G^\alpha \\ &= V_\alpha V^\alpha (\rho_{000} V^\beta)_{;\beta} + V_\alpha \rho_{000} V^\beta (V^\alpha)_{;\beta} + V_\alpha (p g^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha + V_\alpha G^\alpha \\ &= -c^2 (\rho_{000} V^\beta)_{;\beta} + \rho_{000} V^\beta V_\alpha V^\alpha_{;\beta} + V^\alpha P_{;\alpha} \end{aligned}$$

For the heck of it, show by calculation that $V_\alpha V^\alpha_{;\beta} \equiv 0$:

$$\begin{aligned} V_\alpha V^\alpha_{;\beta} &= [\gamma(-c, \underline{v})] [\gamma(c, \underline{v})]_{;\beta} \\ &= V_0 V^0_{;\beta} + V_i V^i_{;\beta} \\ \beta=0 & \\ &= -\gamma c (\gamma c)_{,t} + \gamma v_i (\gamma v_i)_{,t} \\ &= -\gamma c^2 (\gamma)_{,t} + \gamma^2 (\frac{1}{2} v^2)_{,t} + \gamma v^2 \gamma_{,t} = \gamma (v^2 - c^2) \gamma_{,t} + \gamma^2 (\frac{1}{2} v^2)_{,t} \\ \text{now } \gamma_{,t} &= [(1-\beta^2)^{-1/2}]_{,t} = -\frac{1}{2} (1-\beta^2)^{-3/2} (-2\beta) c^{-1} v_{,t} = \gamma^3 c^{-2} v v_{,t} = \gamma^3 c^{-2} (\frac{1}{2} v^2)_{,t} \\ &= \gamma^2 (\gamma/c^2) (\frac{1}{2} v^2)_{,t} \end{aligned}$$

$$\begin{aligned} \text{and } V_\alpha V^\alpha_{;0} &= \gamma (v^2 - c^2) \cdot \gamma^2 (\gamma/c^2) (\frac{1}{2} v^2)_{,t} + \gamma^2 (\frac{1}{2} v^2)_{,t} \\ &= \gamma^2 (\frac{1}{2} v^2)_{,t} [(\gamma^2/c^2) (v^2 - c^2) + 1] = \gamma^2 (\frac{1}{2} v^2)_{,t} [\gamma^2 (\beta^2 - 1) + 1] \\ &= \gamma^2 (\frac{1}{2} v^2)_{,t} [\gamma^2 (-1/\gamma^2) + 1] \equiv 0 \end{aligned}$$

$$\begin{aligned} \beta=j & \\ V_\alpha V^\alpha_{;j} &= V_0 V^0_{;j} + V_i V^i_{;j} \\ &= -\gamma c (\gamma c)_{,j} + \gamma v_i (\gamma v_i)_{,j} = -\gamma c^2 \gamma_{,j} + \gamma^2 (\frac{1}{2} v^2)_{,j} + \gamma v^2 \gamma_{,j} \\ &= \gamma (v^2 - c^2) \gamma_{,j} + \gamma^2 (\frac{1}{2} v^2)_{,j} \\ \text{again } \gamma_{,j} &= [(1-\beta^2)^{-1/2}]_{,j} = -\frac{1}{2} \gamma^3 (-2\beta) c^{-1} v_{i,j} = \gamma^3 c^{-2} v_i v_{i,j} = \gamma^3 c^{-2} (\frac{1}{2} v^2)_{,j} \\ &= \gamma^2 (\gamma/c^2) (\frac{1}{2} v^2)_{,j} \end{aligned}$$

$$\begin{aligned} V_\alpha V^\alpha_{;j} &= \gamma (v^2 - c^2) \cdot \gamma^2 (\gamma/c^2) (\frac{1}{2} v^2)_{,j} + \gamma^2 (\frac{1}{2} v^2)_{,j} \\ &= \gamma^2 (\frac{1}{2} v^2)_{,j} [(\gamma^2/c^2) (v^2 - c^2) + 1] \equiv 0 \end{aligned}$$

Hence we have

$$\begin{aligned} V_\alpha M^{\alpha\beta}_{;\beta} &= -c^2 (\rho_{000} V^\beta)_{;\beta} + V^\alpha P_{;\alpha} \\ &= -[\rho_0 (\vec{a} + \vec{e} + p/\rho_0) V^\beta]_{;\beta} + V^\alpha P_{;\alpha} \end{aligned}$$

$$\begin{aligned} \rho_{000} &= \rho_0 + \rho_0 e \vec{e} + p c^{-2} \\ &= (\rho_0/c^2) (\vec{a} + \vec{e} + p/\rho_0) \\ c^2 \rho_{000} &= \rho_0 (\vec{a} + \vec{e} + p/\rho_0) \end{aligned}$$

$$\begin{aligned}
& \text{Now take } V_\alpha M^{\alpha\beta}_{;\beta} + (c^2 + e + p/\rho_0) [\rho_0 V^\alpha]_{;\alpha} \\
& = - [\rho_0 (c^2 + e + p/\rho_0) V^\beta]_{;\beta} + (c^2 + e + p/\rho_0) (\rho_0 V^\beta)_{;\beta} + V^\alpha p_{,\alpha} \\
& = -(c^2 + e + p/\rho_0) (\rho_0 V^\beta)_{;\beta} + \rho_0 V^\beta (c^2 + e + p/\rho_0)_{;\beta} \\
& \quad + (c^2 + e + p/\rho_0) (\rho_0 V^\beta)_{;\beta} + V^\beta p_{,\beta} \\
& = -\rho_0 V^\beta (e + p/\rho_0)_{;\beta} + V^\beta p_{,\beta} \\
& = -\rho_0 V^\beta e_{;\beta} + \rho_0 V^\beta \rho_0^{-1} p_{,\beta} - \rho_0 V^\beta p (\rho_0^{-1})_{;\beta} + V^\beta p_{,\beta} \\
& = -\rho_0 V^\beta e_{;\beta} + V^\beta p_{,\beta} - \rho_0 p V^\beta (\rho_0^{-1})_{;\beta} + V^\beta p_{,\beta} \\
& = -\rho_0 V^\beta e_{;\beta} - \rho_0 p V^\beta (\rho_0^{-1})_{;\beta}
\end{aligned}$$

But $V^\beta \frac{\partial}{\partial x^\beta} = \frac{D}{Dt}$, hence

$$(94.18) \quad V_\alpha M^{\alpha\beta}_{;\beta} = -\rho_0 [D e / Dt + p D(1/\rho_0) / Dt] = V_\alpha F^\alpha + V_\alpha G^\alpha$$

Now $V_\alpha G^\alpha = \delta(-c, \underline{v}) \cdot (G^0, \underline{G}) = -\delta c G^0 + \delta \underline{v} \cdot \underline{G} = \delta(c G^0 + \underline{v} \cdot \underline{G})$
hence

$$\rho_0 [D e / Dt + p D(1/\rho_0) / Dt] = \delta(c G^0 + \underline{v} \cdot \underline{G})$$

and $Dt = \delta Dt \Rightarrow$

$$(94.19a) \quad \rho_0 [D e / Dt + p D(1/\rho_0) / Dt] = c G^0 - \underline{v} \cdot \underline{G}$$

Finally $G^0 = -c^{-1} (E_{,t} + F^i_{,i})$ (94.2)

$$\underline{G} = R^{j\beta}_{;\beta} = c^{-1} (c^i F^j)_{,t} + P^{ij}_{,j} = c^{-2} F^j_{,t} + P^{ij}_{,j}$$

hence

$$(94.19b) \quad \rho_0 [D e / Dt + p D(1/\rho_0) / Dt] = - [E_{,t} + F^i_{,i}] - \underline{v} \cdot [c^{-2} \underline{E}_{,t} + \underline{\nabla} \cdot \underline{P}]$$

$$(95.1) \quad \frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I}{\partial \mu} = \eta - \chi I$$

now $I/\gamma^3 = I_0/\gamma_0^3$, $\eta/\gamma^2 = \eta_0/\gamma_0^2$ and $\chi\gamma = \chi_0\gamma_0$, hence

$$\frac{1}{c} \frac{\partial}{\partial t} \left(I_0 \frac{\gamma^3}{\gamma_0^3} \right) + \mu \frac{\partial}{\partial r} \left(I_0 \frac{\gamma^3}{\gamma_0^3} \right) + \frac{(1-\mu^2)}{r} \frac{\partial}{\partial \mu} \left(I_0 \frac{\gamma^3}{\gamma_0^3} \right) = \frac{\gamma^2}{\gamma_0^2} (\eta_0 - \chi_0 I_0)$$

and

$$\frac{\partial}{\partial t} \left(I_0 \frac{\gamma^3}{\gamma_0^3} \right) = \frac{\gamma^3}{\gamma_0^3} \frac{\partial I_0}{\partial t} - \frac{3\gamma^3}{\gamma_0^4} I_0 \frac{\partial \gamma_0}{\partial t} = \frac{\gamma^3}{\gamma_0^3} \left[\frac{\partial I_0}{\partial t} - \frac{3}{\gamma_0} \frac{\partial \gamma_0}{\partial t} \right] \text{ etc.}$$

$$\Rightarrow \frac{\gamma^3}{\gamma_0^3} \left[\frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_0}{\partial \mu} \right] - \frac{3\gamma^3}{\gamma_0^4} I_0 \left[\frac{1}{c} \frac{\partial \gamma_0}{\partial t} + \mu \frac{\partial \gamma_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \gamma_0}{\partial \mu} \right] = \frac{\gamma^2}{\gamma_0^2} (\eta_0 - \chi_0 I_0)$$

$$(95.2) \quad \Rightarrow \frac{\gamma}{\gamma_0} \left[\frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_0}{\partial \mu} \right] - \frac{3\gamma I_0}{\gamma_0^2} \left[\frac{1}{c} \frac{\partial \gamma_0}{\partial t} + \mu \frac{\partial \gamma_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial \gamma_0}{\partial \mu} \right] = \eta_0 - \chi_0 I_0$$

Use (89.10) and (89.11) to evaluate derivatives $\frac{\partial \mu_0}{\partial t}$, $\frac{\partial \mu_0}{\partial r}$, $\frac{\partial \gamma_0}{\partial t}$, $\frac{\partial \gamma_0}{\partial r}$, $\frac{\partial \mu_0}{\partial \mu}$, $\frac{\partial \gamma_0}{\partial \mu}$

$$(89.10): \mu_0 = (\mu - \beta) / (1 - \beta\mu) \quad \gamma_0 = \gamma \nu (1 - \beta\mu) \quad (1 - \mu_0^2)^{1/2} = (1 - \mu^2)^{1/2} / \gamma (1 - \beta\mu)$$

$$(89.11): \mu = (\mu_0 + \beta) / (1 + \beta\mu_0) \quad \gamma = \gamma_0 \nu (1 + \beta\mu_0) \quad (1 - \mu^2)^{1/2} = (1 - \mu_0^2)^{1/2} / \gamma (1 + \beta\mu_0)$$

$$\frac{\partial \mu_0}{\partial t} = \frac{1}{(1 - \beta\mu)} \left(-\frac{\partial \beta}{\partial t} \right) + \frac{(\mu - \beta)}{(1 - \beta\mu)^2} \left(+\mu \frac{\partial \beta}{\partial t} \right) = \frac{\partial \beta}{\partial t} \left[\frac{-(1 - \beta\mu) + \mu(\mu - \beta)}{(1 - \beta\mu)^2} \right] = \frac{\partial \beta / \partial t}{(1 + \beta\mu)^2} [\mu^2 - 1]$$

now

$$(1 - \beta\mu) = \frac{\mu - \beta}{\mu_0} = \frac{1}{\mu_0} \left[\frac{\mu_0 + \beta}{1 + \beta\mu_0} - \beta \right] = \frac{1}{\mu_0} \left[\frac{\mu_0 + \beta - \beta - \beta^2 \mu_0}{1 + \beta\mu_0} \right] = \frac{\mu_0 (1 - \beta^2)}{\mu_0 (1 + \beta\mu_0)} = \frac{\gamma^2}{(1 + \beta\mu)}$$

$$\text{hence } (\mu^2 - 1) = (\mu_0^2 - 1) / \gamma^2 (1 + \beta\mu_0)^2$$

$$(\mu^2 - 1) / (1 - \beta\mu)^2 = (\mu_0^2 - 1) \gamma^{-2} (1 + \beta\mu_0)^{-2} (1 + \beta\mu_0)^2 \gamma^4 = \gamma^2 (\mu_0^2 - 1)$$

$$(95.6a) \quad \frac{\partial \mu_0}{\partial t} = \gamma^2 (\mu_0^2 - 1) \frac{\partial \beta}{\partial t} = -\gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial t}$$

an identical analysis (almost) yields

$$(95.7a) \quad \frac{\partial \mu_0}{\partial r} = -\gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial r}$$

$$\frac{\partial \gamma_0}{\partial t} = \frac{\partial}{\partial t} [\gamma \nu (1 - \beta\mu)] = \frac{\partial \gamma}{\partial t} \nu (1 - \beta\mu) + \gamma \nu \left(-\mu \frac{\partial \beta}{\partial t} \right) = \frac{\gamma_0}{\gamma} \frac{\partial \gamma}{\partial t} - \gamma \mu \nu \frac{\partial \beta}{\partial t}$$

$$\text{now } \frac{\partial \gamma}{\partial t} = \frac{\partial}{\partial t} [(1 - \beta^2)^{1/2}] = -\frac{1}{2} (1 - \beta^2)^{-3/2} (-2\beta \frac{\partial \beta}{\partial t}) = \gamma^3 \beta \frac{\partial \beta}{\partial t}$$

$$\text{and } \mu \nu = \gamma_0 \nu (1 + \beta\mu_0) \frac{(\mu_0 + \beta)}{(1 + \beta\mu_0)} = \gamma_0 \nu (\mu_0 + \beta)$$

so

$$\frac{\partial \gamma_0}{\partial t} = \frac{\gamma_0}{\gamma} \left(\gamma^3 \beta \frac{\partial \beta}{\partial t} \right) - \gamma^2 \nu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial t} = \gamma^2 \nu_0 \frac{\partial \beta}{\partial t} [\beta - \mu_0 - \beta] = -\gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial t}$$

$$(95.6b) \quad \frac{\partial \gamma_0}{\partial r} = -\gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial r}$$

the same analysis, replacing $\frac{\partial}{\partial t}$ with $\frac{\partial}{\partial r}$ yields

$$(95.7b) \quad \frac{\partial v_0}{\partial r} = -\gamma^2 \mu_0 v_0 \frac{\partial \beta}{\partial r}$$

$$\frac{\partial \mu}{\partial \mu_0} = \frac{\partial}{\partial \mu_0} \left(\frac{\mu_0 + \beta}{1 + \beta \mu_0} \right) = \frac{1}{1 + \beta \mu_0} - \frac{\mu_0 + \beta}{(1 + \beta \mu_0)^2} \beta = \frac{1}{(1 + \beta \mu_0)^2} [1 + \beta \mu_0 - \beta \mu_0 - \beta^2]$$

$$\text{and } \mu_{0,3} = (1 + \beta^2) / (1 + \beta \mu_0)^2 = \gamma^{-2} (1 + \beta \mu_0)^{-2}$$

$$(95.8) \quad \frac{\partial \mu_0}{\partial \mu} = \left(\frac{\partial \mu}{\partial \mu_0} \right)^{-1} = \gamma^2 (1 + \beta \mu_0)^2$$

$$\left. \frac{\partial v_0}{\partial \mu} \right|_{\text{vt}} = \frac{\partial}{\partial \mu} [\gamma v (1 - \beta \mu)] = -\gamma v \beta = -\gamma \beta [\gamma v_0 (1 + \beta \mu_0)]$$

$$(95.8) \quad \frac{\partial v_0}{\partial \mu} = -\gamma^2 \beta v_0 (1 + \beta \mu_0) \quad \text{Disagrees with D's text.}$$

We use these results in the transfer eqn (95.2) with the derivative

$$\text{expressions: } \frac{\partial}{\partial t} \Big|_{r_{\mu\nu}} = \frac{\partial}{\partial t} \Big|_{r_{\mu_0\nu_0}} + \frac{\partial \mu_0}{\partial t} \Big|_{r_{\mu\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial t} \Big|_{r_{\mu\nu}} \frac{\partial}{\partial v_0}$$

$$\frac{\partial}{\partial r} \Big|_{t_{\mu\nu}} = \frac{\partial}{\partial r} \Big|_{t_{\mu_0\nu_0}} + \frac{\partial \mu_0}{\partial r} \Big|_{t_{\mu\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial r} \Big|_{t_{\mu\nu}} \frac{\partial}{\partial v_0}$$

$$\frac{\partial}{\partial \mu} \Big|_{t_{r\nu}} = \frac{\partial}{\partial \mu} \Big|_{t_{r\nu_0}} + \frac{\partial \mu_0}{\partial \mu} \Big|_{t_{r\nu}} \frac{\partial}{\partial \mu_0} + \frac{\partial v_0}{\partial \mu} \Big|_{t_{r\nu}} \frac{\partial}{\partial v_0}$$

of the results just derived, we have:

$$\frac{\partial \mu_0}{\partial t} = -\gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial t} \quad \frac{\partial v_0}{\partial t} = -\gamma^2 v_0 \mu_0 \left(\frac{\partial \beta}{\partial t} \right)$$

$$\frac{\partial \mu_0}{\partial r} = -\gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial r} \quad \frac{\partial v_0}{\partial r} = -\gamma^2 v_0 \mu_0 \left(\frac{\partial \beta}{\partial r} \right)$$

$$\frac{\partial \mu_0}{\partial \mu} = \gamma^2 (1 + \beta \mu_0)^2 \quad \frac{\partial v_0}{\partial \mu} = -\beta \gamma^2 (1 + \beta \mu_0) v_0$$

$$\text{need } v = \gamma v_0 (1 + \beta \mu_0) \text{ or } (v/v_0) = \gamma (1 + \beta \mu_0)$$

$$1 = \frac{\partial I_0}{\partial t} \Big|_{r_{\mu_0\nu_0}} - \gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial t} \frac{\partial I_0}{\partial \mu_0} - \gamma^2 v_0 \mu_0 \frac{\partial \beta}{\partial t} \frac{\partial I_0}{\partial v_0}$$

$$v = \frac{\mu_0 + \beta}{(1 + \beta \mu_0)} \left\{ \frac{\partial I_0}{\partial r} \Big|_{r_{\mu_0\nu_0}} - \gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial r} \frac{\partial I_0}{\partial \mu_0} - \gamma^2 v_0 \mu_0 \frac{\partial \beta}{\partial r} \frac{\partial I_0}{\partial v_0} \right\}$$

$$\frac{\partial \mu}{\partial r_{\mu\nu}} = \frac{(1 - \mu_0^2)}{\gamma^2 (1 + \beta \mu_0)^2} \left\{ \gamma^2 (1 + \beta \mu_0)^2 \frac{\partial I_0}{\partial \mu_0} - \beta \gamma^2 (1 + \beta \mu_0) v_0 \frac{\partial I_0}{\partial v_0} \right\}$$

$$\begin{aligned}
& \text{and } \frac{v}{v_0} \left[\frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial I_0}{\partial \mu} \right] \\
&= \gamma(1+\beta\mu_0) \left\{ \frac{1}{c} \frac{\partial I_0}{\partial t} + \frac{(\mu_0+\beta)}{(1+\beta\mu_0)} \frac{\partial I_0}{\partial r} \right\} \\
&+ \gamma(1+\beta\mu_0) \left\{ -\frac{\gamma^2(1-\mu_0^2)}{c} \frac{\partial \beta}{\partial t} - \frac{(\mu_0+\beta)}{(1+\beta\mu_0)} \gamma^2(1-\mu_0^2) \frac{\partial \beta}{\partial r} + \frac{(1-\mu_0^2)}{\gamma^2(1+\beta\mu_0)^2} \frac{\partial \beta}{\partial \mu} \right\} \frac{\partial I_0}{\partial \mu_0} \\
&+ \gamma(1+\beta\mu_0) \left\{ -\frac{\gamma^2}{c} \gamma_0 \mu_0 \frac{\partial \beta}{\partial t} - \frac{(\mu_0+\beta)}{(1+\beta\mu_0)} \gamma^2 \gamma_0 \mu_0 \frac{\partial \beta}{\partial r} - \frac{(1-\mu_0^2)}{\gamma^2(1+\beta\mu_0)^2} \beta \frac{\partial \beta}{\partial \mu} \right\} \frac{\partial I_0}{\partial \nu_0} \\
&= \frac{\gamma}{c} (1+\beta\mu_0) \frac{\partial I_0}{\partial t} + \gamma(\mu_0+\beta) \frac{\partial I_0}{\partial r} \\
&+ \gamma(1-\mu_0^2) \left\{ -\frac{\gamma^2}{c} (1+\beta\mu_0) \frac{\partial \beta}{\partial t} - \gamma^2(\mu_0+\beta) \frac{\partial \beta}{\partial r} + \frac{(1+\beta\mu_0)}{r} \right\} \frac{\partial I_0}{\partial \mu_0} \\
&+ \gamma \gamma_0 \left\{ -\frac{\gamma^2}{c} (1+\beta\mu_0) \mu_0 \frac{\partial \beta}{\partial t} - \gamma^2(\mu_0+\beta) \mu_0 \frac{\partial \beta}{\partial r} - \frac{(1-\mu_0^2)}{r} \beta \right\} \frac{\partial I_0}{\partial \nu_0} \\
&= \frac{\gamma}{c} (1+\beta\mu_0) \frac{\partial I_0}{\partial t} + \gamma(\mu_0+\beta) \frac{\partial I_0}{\partial r} + \gamma(1-\mu_0^2) \left\{ \frac{(1+\beta\mu_0)}{r} - \frac{\gamma^2(1+\beta\mu_0)}{c} \frac{\partial \beta}{\partial t} - \gamma^2(\mu_0+\beta) \frac{\partial \beta}{\partial r} \right\} \frac{\partial I_0}{\partial \mu_0} \\
&- \gamma \gamma_0 \left\{ \frac{(1-\mu_0^2)}{r} \beta + \frac{\gamma^2(1+\beta\mu_0) \mu_0}{c} \frac{\partial \beta}{\partial t} + \gamma^2(\mu_0+\beta) \mu_0 \frac{\partial \beta}{\partial r} \right\} \frac{\partial I_0}{\partial \nu_0}
\end{aligned}$$

Next the term: $-3I_0(v/v_0^2) \left[\frac{1}{c} \frac{\partial v_0}{\partial t} + \mu \frac{\partial v_0}{\partial r} + \frac{(1-\mu^2)}{r} \frac{\partial v_0}{\partial \mu} \right]$

$$\begin{aligned}
[] &= \frac{1}{c} (-\gamma^2 \mu_0 \gamma_0 \frac{\partial \beta}{\partial t}) + \frac{(\mu_0+\beta)}{(1+\beta\mu_0)} (-\gamma^2 \gamma_0 \mu_0 \frac{\partial \beta}{\partial r}) + \frac{(1-\mu_0^2)}{\gamma^2(1+\beta\mu_0)^2} (-3\gamma^2(1+\beta\mu_0) \gamma_0) \\
&= -\frac{\gamma_0}{c} \left\{ \gamma^2 \mu_0 \frac{\partial \beta}{\partial t} + \frac{\gamma^2(\mu_0+\beta) \mu_0}{1+\beta\mu_0} \frac{\partial \beta}{\partial r} + \frac{\beta \gamma (1-\mu_0^2)}{(1+\beta\mu_0)r} \right\}
\end{aligned}$$

$$\begin{aligned}
-3I_0 \frac{\gamma(1+\beta\mu_0)}{\gamma_0} (-\gamma_0) \left\{ \right\} &= +3I_0 \gamma \left\{ \frac{\gamma^2 \mu_0}{c} (1+\beta\mu_0) \frac{\partial \beta}{\partial t} + \gamma^2(\mu_0+\beta) \mu_0 \frac{\partial \beta}{\partial r} + \frac{\beta}{r} (1-\mu_0^2) \right\} \\
&= 3\gamma I_0 \left\{ \frac{\beta(1-\mu_0^2)}{r} + \frac{\gamma^2}{c} \mu_0(1+\beta\mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0(\mu_0+\beta) \frac{\partial \beta}{\partial r} \right\}
\end{aligned}$$

The whole eqn becomes:

$$\begin{aligned}
&\frac{\gamma}{c} (1+\beta\mu_0) \frac{\partial I_0}{\partial t} + \gamma(\mu_0+\beta) \frac{\partial I_0}{\partial r} + \gamma(1-\mu_0^2) \left\{ \frac{(1+\beta\mu_0)}{r} - \frac{\gamma^2}{c} (1+\beta\mu_0) \frac{\partial \beta}{\partial t} - \gamma^2(\mu_0+\beta) \frac{\partial \beta}{\partial r} \right\} \frac{\partial I_0}{\partial \mu_0} \\
&- \gamma \gamma_0 \left\{ \frac{(1-\mu_0^2)}{r} \beta + \frac{\gamma^2}{c} (1+\beta\mu_0) \mu_0 \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0(\mu_0+\beta) \frac{\partial \beta}{\partial r} \right\} \frac{\partial I_0}{\partial \nu_0} \\
&+ 3\gamma \left\{ \frac{\beta(1-\mu_0^2)}{r} + \frac{\gamma^2}{c} \mu_0(1+\beta\mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0(\mu_0+\beta) \frac{\partial \beta}{\partial r} \right\} I_0 = \gamma_0 - \gamma_0 I_0
\end{aligned}$$

(95.9)

$$3\gamma \left[\frac{\beta(1-\mu_0^2)}{\gamma} + \frac{\gamma^2}{c} \mu_0^2 (1+\beta\mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right] I_0$$

$$- I_0 \frac{2}{\partial \mu_0} \left\{ \gamma (1-\mu_0^2) \left[\frac{1+\beta\mu_0}{\gamma} - \gamma^2 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} - \frac{\gamma^2}{c} (1+\beta\mu_0) \frac{\partial \beta}{\partial t} \right] \right\}$$

sign error $\oplus I_0 \frac{2}{\partial \mu_0} \left\{ \gamma \gamma_0 \left[\frac{\beta(1-\mu_0^2)}{\gamma} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} + \frac{\gamma^2}{c} \mu_0 (1+\beta\mu_0) \frac{\partial \beta}{\partial t} \right] \right\}$

$$\frac{1}{I_0} = 3\gamma \frac{\beta(1-\mu_0^2)}{\gamma} + 3\gamma \cdot \frac{\gamma^2}{c} \mu_0^2 (1+\beta\mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r}$$

$$+ 2\gamma \frac{\mu_0(1+\beta\mu_0)}{\gamma} - 2\gamma \mu_0 \frac{\gamma^2}{c} (1+\beta\mu_0) \frac{\partial \beta}{\partial t} - 2\gamma \mu_0 \gamma^2 (\mu_0 + \beta) \frac{\partial \beta}{\partial r}$$

$$= \gamma (1-\mu_0^2) \frac{\beta}{\gamma} + \gamma (1-\mu_0^2) \frac{\gamma^2}{c} \beta \frac{\partial \beta}{\partial t} + \gamma (1-\mu_0^2) \gamma^2 \frac{\partial \beta}{\partial r}$$

$$\oplus \gamma \frac{\beta(1-\mu_0^2)}{\gamma} \oplus \gamma \cdot \frac{\gamma^2}{c} \mu_0 (1+\beta\mu_0) \frac{\partial \beta}{\partial t} \oplus \gamma \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r}$$

$$= \frac{\gamma}{\gamma} \left\{ 3\beta(1-\mu_0^2) + 2\mu_0(1+\beta\mu_0) - \gamma(1-\mu_0^2)\beta \oplus \beta(1-\mu_0^2) \right\}$$

$$+ \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ 3\mu_0(1+\beta\mu_0) - 2\mu_0(1+\beta\mu_0) + (1-\mu_0^2)\beta \oplus \mu_0(1+\beta\mu_0) \right\}$$

$$+ \gamma^3 \frac{\partial \beta}{\partial r} \left\{ 3\mu_0(\mu_0 + \beta) - 2\mu_0(\mu_0 + \beta) + (1-\mu_0^2) \oplus \mu_0(\mu_0 + \beta) \right\}$$

$$= \frac{\gamma}{\gamma} \left\{ \beta(1-\mu_0^2) + 2\mu_0(1+\beta\mu_0) \right\} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ \mu_0(1+\beta\mu_0) + (1-\mu_0^2)\beta - \mu_0(1+\beta\mu_0) \right\} + \gamma^3 \frac{\partial \beta}{\partial r} \left\{ 1-\mu_0^2 \right\}$$

$$= \frac{\gamma}{\gamma} \left\{ \beta - \beta\mu_0^2 + 2\mu_0 + 2\beta\mu_0^2 \right\} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ \beta(1-\mu_0^2) + \gamma^3 \frac{\partial \beta}{\partial r} (1-\mu_0^2) \right\}$$

line 4 of new eqn (95.9)

$$\begin{aligned} &= \frac{\gamma}{r} \left\{ 3\beta(1-\mu^2) + 2\mu_0(1+\beta\mu_0) \right\} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ 2\mu_0(1+\beta\mu_0) + \beta(1-\mu_0^2) \right\} + \frac{\gamma^3}{\partial r} \left\{ 2\mu_0(\mu_0+\beta) + (1-\mu_0^2) \right\} \\ &= \frac{\gamma}{r} \left\{ 3\beta - 3\beta\mu_0^2 + 2\mu_0 + 2\beta\mu_0^2 \right\} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ 2\mu_0 + 2\beta\mu_0^2 + \beta - \beta\mu_0^2 \right\} + \frac{\gamma^3}{\partial r} \left\{ 2\mu_0^2 + 2\beta\mu_0 + 1 - \mu_0^2 \right\} \\ &= \frac{\gamma}{r} \left\{ \beta(3-\mu_0^2) + 2\mu_0 \right\} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ 2\mu_0 + \beta(\mu_0^2 + 1) \right\} + \frac{\gamma^3}{\partial r} \left\{ 2\beta\mu_0 + \mu_0^2 + 1 \right\} \end{aligned}$$

$$(a) \quad \frac{\gamma}{c} \int (1+\beta\mu_0) \frac{\partial I_0}{\partial t} \frac{d\omega_0}{4\pi} + \gamma \int (\mu_0 + \beta) \frac{\partial I_0}{\partial r} \frac{d\omega_0}{4\pi} + \gamma \int (1-\mu_0^2) \frac{(1+\beta\mu_0)}{\gamma} \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} - \frac{\gamma^3}{c} \int (1-\mu_0^2)(1+\beta\mu_0) \frac{\partial \beta}{\partial t} \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi}$$

$$- \gamma^3 \frac{\partial \beta}{\partial r} \int (1-\mu_0^2)(\mu_0 + \beta) \frac{\partial I_0}{\partial \mu_0} - \frac{\gamma v_0}{\gamma} \int (1-\mu_0^2) \frac{\partial I_0}{\partial v_0} \frac{d\omega_0}{4\pi} - \frac{\partial^3 v_0}{c} \frac{\partial \beta}{\partial t} \int \mu_0 (1+\beta\mu_0) \frac{\partial I_0}{\partial v_0} \frac{d\omega_0}{4\pi} - \gamma^3 v_0 \frac{\partial \beta}{\partial r} \int \mu_0 (\mu_0 + \beta) \frac{\partial I_0}{\partial v_0} \frac{d\omega_0}{4\pi}$$

$$+ \frac{3\gamma\beta}{\gamma} \int (1-\mu_0^2) I_0 \frac{d\omega_0}{4\pi} + \frac{3\gamma^3}{c} \frac{\partial \beta}{\partial t} \int \mu_0 (1+\beta\mu_0) I_0 \frac{d\omega_0}{4\pi} + 3\gamma^3 \frac{\partial \beta}{\partial r} \int \mu_0 (\mu_0 + \beta) I_0 \frac{d\omega_0}{4\pi} = \int (\gamma_0 - \gamma_0 I_0) \frac{d\omega_0}{4\pi}$$

$$(b) \quad = \frac{\gamma}{c} \left[\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right] + \gamma \left[\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right] + \frac{\gamma}{\gamma} (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} - \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \int (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi}$$

$$- \gamma^3 \frac{\partial \beta}{\partial r} \int (\mu_0 - \mu_0^3 + \beta - \beta\mu_0^2) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} - \frac{\gamma v_0}{\gamma} \left(\frac{\partial J_0}{\partial v_0} - \frac{\partial K_0}{\partial v_0} \right) - \frac{\gamma^3 v_0}{c} \frac{\partial \beta}{\partial t} \left(\frac{\partial H_0}{\partial v_0} + \beta \frac{\partial K_0}{\partial v_0} \right) - \gamma^3 v_0 \frac{\partial \beta}{\partial r} \left(\frac{\partial K_0}{\partial v_0} + \beta \frac{\partial H_0}{\partial v_0} \right)$$

$$+ \frac{3\gamma\beta}{\gamma} (J_0 - K_0) + \frac{3\gamma^3}{c} \frac{\partial \beta}{\partial t} (H_0 + \beta K_0) + 3\gamma^3 \frac{\partial \beta}{\partial r} (\beta H_0 + K_0) = \gamma_0 - \gamma_0 J_0$$

Evaluate $\int (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) \frac{\partial I_0}{\partial \mu_0} d\mu_0 = \int_{\mu_0=0}^1 \left\{ (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) I_0 \right\} d\mu_0$
 $- \int I_0 \frac{\partial}{\partial \mu_0} (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) d\mu_0$

$$= \left[I_0 (1-\mu_0^2 + \beta\mu_0 - \beta\mu_0^3) \right]_{\mu_0=0}^1 - \int I_0 [-2\mu_0 + \beta - 3\beta\mu_0^2] d\mu_0$$

$$= I_0 (\mu_0=1) (1-1+\beta-\beta) - I_0 (-1) [-1+\beta(1+1)] + \int I_0 (2\mu_0 - \beta + 3\beta\mu_0^2) d\mu_0$$

$$= 0 + 0 + (2H_0 - \beta J_0 + 3\beta K_0) = -(\beta J_0 + 2H_0 - 3\beta K_0)$$

(c)

Similarly $\int (\mu_0 - \mu_0^3 + \beta - \beta\mu_0^2) \frac{\partial I_0}{\partial \mu_0} d\mu_0 = 0 - \int I_0 \frac{\partial}{\partial \mu_0} (\mu_0 - \mu_0^3 + \beta - \beta\mu_0^2) d\mu_0$
 $= - \int I_0 (1 - 3\mu_0^2 - 2\beta\mu_0) d\mu_0 = \int I_0 (2\beta\mu_0 + 3\mu_0^2 - 1) d\mu_0$
 $= (-J_0 + 2\beta H_0 + 3K_0) = -(J_0 - 2\beta H_0 - 3K_0)$

(d)

Thus the three terms in $\partial I_0 / \partial \mu_0$ integrate to:

$$(e) \quad \left(-\frac{\gamma}{\gamma} + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \right) (\beta J_0 - 2H_0 - 3\beta K_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 - 2\beta H_0 - 3K_0)$$

Add these terms to the 3rd line of (b):

$$-\frac{\gamma}{\gamma} (\beta J_0 - 2H_0 - 3\beta K_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (\beta J_0 - 2H_0 - 3\beta K_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 - 2\beta H_0 - 3K_0)$$

$$+ \frac{\gamma}{\gamma} (3\beta J_0 - 3\beta K_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (3H_0 + 3\beta K_0) + \gamma^3 \frac{\partial \beta}{\partial r} (3\beta H_0 + 3K_0)$$

$$= \frac{\gamma}{\gamma} (2\beta J_0 + 2H_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (\beta J_0 + H_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 + \beta H_0)$$

Thus the entire zeroth moment equation becomes:

(95.11)

$$\frac{\gamma}{c} \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \gamma \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) - \frac{\gamma v_0 \beta}{r} \left(\frac{\partial J_0}{\partial v_0} - \frac{\partial K_0}{\partial v_0} \right) - \frac{\gamma^3 v_0}{c} \frac{\partial \beta}{\partial t} \left(\frac{\partial H_0}{\partial v_0} + \beta \frac{\partial K_0}{\partial v_0} \right) - \frac{\gamma^3 v_0}{c} \frac{\partial \beta}{\partial r} \left(\frac{\partial K_0}{\partial v_0} + \beta \frac{\partial H_0}{\partial v_0} \right) + \frac{2\gamma}{r} (\beta J_0 + H_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (\beta J_0 + H_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 + \beta H_0) = \eta_0 - \chi_0 J_0$$

Now start again and integrate against $\mu_0 d\omega_0/4\pi$:

$$\begin{aligned} & \frac{\gamma}{c} \int (\mu_0 + \beta \mu_0^2) \frac{\partial I_0}{\partial t} \frac{d\omega_0}{4\pi} + \gamma \int (\mu_0^2 + \beta \mu_0) \frac{\partial I_0}{\partial r} \frac{d\omega_0}{4\pi} + \gamma \int (\mu_0 - \mu_0^3) \left\{ \frac{1 + \beta \mu_0}{r} - \frac{\partial^2}{c} (1 + \beta \mu_0) \frac{\partial \beta}{\partial t} - \gamma^2 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right\} \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} \\ & - \gamma v_0 \int \left\{ \frac{(1 - \mu_0^2) \beta}{r} + \frac{\partial^2}{c} (1 + \beta \mu_0) \mu_0 \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right\} \mu_0 \frac{\partial I_0}{\partial v_0} \\ & + 3 \gamma \int \left\{ \beta \frac{(1 - \mu_0^2)}{r} + \frac{\partial^2}{c} \mu_0 (1 + \beta \mu_0) \frac{\partial \beta}{\partial t} + \gamma^2 \mu_0 (\mu_0 + \beta) \frac{\partial \beta}{\partial r} \right\} \mu_0 I_0 \frac{d\omega_0}{4\pi} = \int \mu_0 (\eta_0 - \chi_0 I_0) \frac{d\omega_0}{4\pi} \\ & = \frac{\gamma}{c} \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) + \gamma \left(\beta \frac{\partial H_0}{\partial r} + \frac{\partial K_0}{\partial r} \right) - \gamma v_0 \left\{ \frac{\beta}{r} \int (\mu_0 - \mu_0^3) \frac{\partial I_0}{\partial v_0} \frac{d\omega_0}{4\pi} + \frac{\gamma}{c} \int (\mu_0^2 + \beta \mu_0^3) \frac{\partial \beta}{\partial t} \frac{\partial I_0}{\partial v_0} \frac{d\omega_0}{4\pi} + \frac{\partial^2 \beta}{\partial r} \int (\mu_0^3 + \beta \mu_0^4) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} \right\} \\ & + \frac{\gamma}{r} \int (\mu_0 - \mu_0^3) (1 + \beta \mu_0) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} - \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \int (\mu_0 - \mu_0^3) (1 + \beta \mu_0) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} - \frac{\gamma^3}{\partial r} \int (\mu_0 - \mu_0^3) (\mu_0 + \beta) \frac{\partial I_0}{\partial \mu_0} \frac{d\omega_0}{4\pi} \\ & + 3 \gamma \frac{\beta}{r} \int (\mu_0 - \mu_0^3) I_0 \frac{d\omega_0}{4\pi} + \frac{3 \gamma^3}{c} \frac{\partial \beta}{\partial t} \int (\mu_0^2 + 3 \mu_0^3) I_0 \frac{d\omega_0}{4\pi} + 3 \gamma^3 \frac{\partial \beta}{\partial r} \int (\mu_0^3 + 3 \mu_0^2) I_0 \frac{d\omega_0}{4\pi} = (\eta_0 - \chi_0 I_0) \mu_0 \frac{d\omega_0}{4\pi} \end{aligned}$$

Again we must integrate terms in $\frac{\partial I_0}{\partial \mu_0}$ by parts:

$$\begin{aligned} & \int (\mu_0 - \mu_0^3) (1 + \beta \mu_0) \frac{\partial I_0}{\partial \mu_0} d\mu_0 = \int (\mu_0 - \mu_0^3 + \beta \mu_0^2 - 3 \mu_0^4) \frac{\partial I_0}{\partial \mu_0} d\mu_0 \\ & = \int \frac{\partial}{\partial \mu_0} \left\{ (\mu_0 - \mu_0^3 + \beta \mu_0^2 - \beta \mu_0^4) I_0 \right\} d\mu_0 - \int I_0 \frac{\partial}{\partial \mu_0} (\mu_0 + \beta \mu_0^2 - \mu_0^3 - \beta \mu_0^4) d\mu_0 \\ & = \left[(\mu_0 - \mu_0^3 + \beta \mu_0^2 - \beta \mu_0^4) I_0 \right]_{-1}^{+1} - \int I_0 (1 + 2\beta \mu_0 - 3\mu_0^2 - 4\beta \mu_0^3) d\mu_0 \\ & = 0 - (J_0 + 2\beta H_0 - 3K_0 - 4\beta N_0) = (-J_0 + 3K_0 - 2\beta H_0 + 4\beta N_0) \end{aligned}$$

$$\begin{aligned} \text{and } & \int (\mu_0 - \mu_0^3) (\mu_0 + \beta) \frac{\partial I_0}{\partial \mu_0} d\mu_0 = \int (\mu_0^2 - \mu_0^4 + \beta \mu_0 - \beta \mu_0^3) \frac{\partial I_0}{\partial \mu_0} d\mu_0 \\ & = - \int I_0 \frac{\partial}{\partial \mu_0} (\mu_0^2 - \mu_0^4 + \beta \mu_0 - \beta \mu_0^3) d\mu_0 = - \int I_0 [2\mu_0 - 4\mu_0^3 + \beta - 3\beta \mu_0^2] d\mu_0 \\ & = -(2H_0 - 4N_0 + \beta J_0 - 3\beta K_0) = (-\beta J_0 - 2H_0 + 3\beta K_0 + 4N_0) \end{aligned}$$

The terms in $\frac{\partial I_0}{\partial \mu_0}$ then become:

$$\frac{\gamma}{r} (-J_0 + 3K_0 - 2\beta H_0 + 4\beta N_0) - \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (-J_0 + 3K_0 - 2\beta H_0 + 4\beta N_0) - \frac{\gamma^3}{\partial r} (-\beta J_0 - 2H_0 + 3\beta K_0 + 4N_0)$$

and the terms in I_0 are:

$$\frac{\gamma}{r} (3\beta H_0 - 3\beta N_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (3K_0 + 3\beta N_0) + \gamma^3 \frac{\partial \beta}{\partial r} (3\beta K_0 + 3N_0)$$

Adding these two sets of terms we obtain:

$$\frac{\gamma}{c} (-J_0 + \beta H_0 + 3K_0 + \beta N_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (J_0 + 2\beta H_0 - \beta N_0) + \gamma^3 \frac{\partial \beta}{\partial r} (\beta J_0 + 2H_0 - N_0)$$

Thus the 1st moment equation becomes

$$(95.12) \quad \frac{\gamma}{c} \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) + \gamma \left(\beta \frac{\partial H_0}{\partial r} + \frac{\partial K_0}{\partial r} \right) - \gamma v_0 \left\{ \frac{\gamma}{c} \left(\frac{\partial H_0}{\partial v_0} - \frac{\partial N_0}{\partial v_0} \right) + \frac{\gamma^2}{c} \frac{\partial \beta}{\partial t} \left(\frac{\partial K_0}{\partial v_0} + \beta \frac{\partial N_0}{\partial v_0} \right) + \gamma^2 \frac{\partial \beta}{\partial r} \left(\beta \frac{\partial K_0}{\partial v_0} + \frac{\partial N_0}{\partial v_0} \right) \right. \\ \left. + \frac{\gamma}{r} (-J_0 + \beta H_0 + 3K_0 + \beta N_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (J_0 + 2\beta H_0 - \beta N_0) + \gamma^3 \frac{\partial \beta}{\partial r} (\beta J_0 + 2H_0 - N_0) \right\} = -\chi_0 H_0$$

Integrate over dv_0 to obtain frequency-integrated moment equations:
(95.11) becomes:

$$\frac{\gamma}{c} \int \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) dv_0 + \gamma \int \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) dv_0 - \frac{\gamma \beta}{r} \int \left(\frac{\partial K_0}{\partial v_0} - \frac{\partial N_0}{\partial v_0} \right) v_0 dv_0 - \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \int \left(\frac{\partial H_0}{\partial v_0} + \beta \frac{\partial K_0}{\partial v_0} \right) v_0 dv_0 \\ - \gamma^3 \frac{\partial \beta}{\partial r} \int \left(\frac{\partial K_0}{\partial v_0} + \beta \frac{\partial N_0}{\partial v_0} \right) v_0 dv_0 + 2\frac{\gamma}{r} \int (\beta J_0 + H_0) dv_0 + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \int (\beta J_0 + H_0) dv_0 + \gamma^3 \frac{\partial \beta}{\partial r} \int (J_0 + \beta H_0) dv_0 \\ = \int (\dot{M}_0 - \chi_0 J_0) dv_0$$

Using now the symbols $J_0^f, K_0, H_0 + N_0$ to mean frequency-integrated quantities, we have

$$\frac{\gamma}{c} \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \gamma \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \frac{2\gamma}{r} (\beta J_0 + H_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (\beta J_0 + H_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 + \beta H_0) \\ - \frac{\gamma \beta}{r} \int v_0 dv_0 \left(\frac{\partial J_0(v_0)}{\partial v_0} - \frac{\partial K_0(v_0)}{\partial v_0} \right) - \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \int v_0 dv_0 \left(\frac{\partial H_0(v_0)}{\partial v_0} + \beta \frac{\partial K_0(v_0)}{\partial v_0} \right) - \gamma^3 \frac{\partial \beta}{\partial r} \int v_0 dv_0 \left(\frac{\partial K_0(v_0)}{\partial v_0} + \beta \frac{\partial H_0(v_0)}{\partial v_0} \right) \\ = \int (\dot{M}_0(v_0) - \chi_0(v_0) J_0(v_0)) dv_0$$

Integrating the terms in $\partial/\partial v_0$ by parts we obtain:

$$\int v_0 dv_0 \left(\frac{\partial J_0(v_0)}{\partial v_0} - \frac{\partial K_0(v_0)}{\partial v_0} \right) = \int \frac{2}{\partial v_0} [v_0 J_0 - v_0 K_0] dv_0 - \int (J_0 + K_0) dv_0 \\ = [v_0 J_0 - v_0 K_0]_{v_0=0}^{\infty} - (J_0 - K_0)$$

As $v_0 \rightarrow \infty$, J_0 and $K_0 \rightarrow 0$ exponentially, while at $v_0=0$, $J_0 + K_0$ are finite hence $\int = 0 - (J_0 + K_0)$

and

$$-\frac{\gamma \beta}{r} \int v_0 dv_0 \left(\frac{\partial J_0(v_0)}{\partial v_0} - \frac{\partial K_0(v_0)}{\partial v_0} \right) = +\frac{\gamma \beta}{r} (J_0 - K_0)$$

A similar analysis gives:

$$-\gamma^3 \frac{\partial \beta}{\partial t} \int \left(\frac{\partial H_0(v_0)}{\partial v_0} + \beta \frac{\partial K_0(v_0)}{\partial v_0} \right) v_0 dv_0 = + \gamma^3 \frac{\partial \beta}{\partial t} \int (H_0(v_0) + \beta K_0(v_0)) dv_0 = \gamma^3 \frac{\partial \beta}{\partial t} (H_0 + \beta K_0)$$

and

$$-\gamma^3 \frac{\partial \beta}{\partial r} \int \left(\frac{\partial K_0(v_0)}{\partial v_0} + \beta \frac{\partial H_0(v_0)}{\partial v_0} \right) v_0 dv_0 = \gamma^3 \frac{\partial \beta}{\partial r} \int (K_0(v_0) + \beta H_0(v_0)) dv_0 = \gamma^3 \frac{\partial \beta}{\partial r} (K_0 + \beta H_0)$$

The v_0 -integrated zeroth moment eqn becomes:

$$\begin{aligned} & \frac{\gamma}{c} \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \gamma \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \frac{\gamma}{r} (\beta J_0 + H_0) + \gamma^3 \frac{\partial \beta}{\partial t} (\beta J_0 + H_0) + \gamma^3 \frac{\partial \beta}{\partial r} (J_0 + \beta H_0) \\ & + \frac{\gamma}{r} (\beta J_0 - \beta K_0) + \gamma^3 \frac{\partial \beta}{\partial t} (H_0 + \beta K_0) + \gamma^3 \frac{\partial \beta}{\partial r} (K_0 + \beta H_0) \\ & = \int (\eta_0(v_0) - \chi_0(v_0) J_0(v_0)) dv_0 \end{aligned}$$

(9.13)

$$\begin{aligned} & \frac{\gamma}{c} \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \gamma \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \frac{\gamma}{r} [3\beta J_0 + 2H_0 - \beta K_0] + \gamma^3 \frac{\partial \beta}{\partial t} [\beta J_0 + 2H_0 + \beta K_0] \\ & + \gamma^3 \frac{\partial \beta}{\partial r} [J_0 + 2\beta H_0 + K_0] = \int_0^\infty [\eta_0(v_0) - \chi_0(v_0) J_0(v_0)] dv_0 \end{aligned}$$

Now integrate the first moment equation over frequency:

$$\begin{aligned} & \frac{\gamma}{c} \int \left(\frac{\partial H_0(v_0)}{\partial t} + \beta \frac{\partial K_0(v_0)}{\partial t} \right) dv_0 + \gamma \int \left(\beta \frac{\partial H_0(v_0)}{\partial r} + \frac{\partial K_0(v_0)}{\partial r} \right) dv_0 \\ & + \frac{\gamma}{r} \int [-J_0(v_0) + \beta H_0(v_0) + 3K_0(v_0) + \beta N_0(v_0)] dv_0 + \gamma^3 \frac{\partial \beta}{\partial t} \int [J_0(v_0) + 2\beta H_0(v_0) - \beta N_0(v_0)] dv_0 \\ & + \gamma^3 \frac{\partial \beta}{\partial r} \int [3J_0(v_0) + 2H_0(v_0) - N_0(v_0)] dv_0 \\ & - \gamma \beta \int \left(\frac{\partial H_0(v_0)}{\partial v_0} - \frac{\partial N_0(v_0)}{\partial v_0} \right) v_0 dv_0 - \gamma^3 \frac{\partial \beta}{\partial t} \int \left[\frac{\partial K_0(v_0)}{\partial v_0} + \beta \frac{\partial H_0(v_0)}{\partial v_0} \right] v_0 dv_0 - \gamma^3 \frac{\partial \beta}{\partial r} \int \left[\beta \frac{\partial K_0(v_0)}{\partial v_0} + \frac{\partial N_0(v_0)}{\partial v_0} \right] v_0 dv_0 \\ & = - \int \chi_0(v_0) H_0(v_0) dv_0 \end{aligned}$$

The terms in $\partial/\partial v_0$ become:

$$\begin{aligned} & -(\gamma\beta/r) \left\{ - \int (H_0(v_0) - N_0(v_0)) dv_0 \right\} = \frac{\gamma}{r} (\beta H_0 - \beta N_0) \\ & - \gamma^3 \frac{\partial \beta}{\partial t} \left\{ - \int (K_0(v_0) + \beta N_0(v_0)) dv_0 \right\} = \gamma^3 \frac{\partial \beta}{\partial t} (K_0 + \beta N_0) \\ & - \gamma^3 \frac{\partial \beta}{\partial r} \left\{ - \int [\beta K_0(v_0) + N_0(v_0)] dv_0 \right\} = \gamma^3 \frac{\partial \beta}{\partial r} (\beta K_0 + N_0) \end{aligned}$$

and the equation becomes

$$\begin{aligned} & \frac{\gamma}{c} \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) + \gamma \left(\beta \frac{\partial H_0}{\partial r} + \frac{\partial K_0}{\partial r} \right) + \frac{\gamma}{r} (-J_0 + \beta H_0 + 3K_0 + \beta N_0) + \frac{\gamma}{r} (\beta H_0 - \beta N_0) \\ & + \gamma^3 \frac{\partial \beta}{\partial t} (J_0 + 2\beta H_0 - \beta N_0) + \gamma^3 \frac{\partial \beta}{\partial t} (K_0 + \beta N_0) + \gamma^3 \frac{\partial \beta}{\partial r} (\beta J_0 + 2H_0 - N_0) + \gamma^3 \frac{\partial \beta}{\partial r} (\beta K_0 + N_0) \\ & = - \int \chi_0(v_0) H_0(v_0) dv_0 \end{aligned}$$

$$(9.14) \quad \frac{\gamma}{c} \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) + \gamma \left(\beta \frac{\partial H_0}{\partial r} + \frac{\partial K_0}{\partial r} \right) + \frac{\gamma}{r} (-J_0 + 2\beta H_0 + 3K_0) + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} (J_0 + K_0 + 2\beta H_0) + \gamma^3 \frac{\partial \beta}{\partial r} (2H_0 + \beta J_0 + \beta K_0) = - \int \gamma_0(v_0) H_0(v_0) dv_0$$

Now check these by using inertial-frame moment equations and transforming $(J, H, K) \rightarrow (J_0, H_0, K_0)$ via eqns (9.13)-(9.15)

$$c^{-1} \frac{\partial J}{\partial t} + \frac{\partial H}{\partial r} + \frac{2H}{r} = - \left(\frac{c}{4\pi} \right) G^0$$

$$c^{-1} \frac{\partial H}{\partial t} + \frac{\partial K}{\partial r} + \frac{3K-J}{r} = - \left(\frac{c}{4\pi} \right) G^1$$

Write (9.13)-(9.15) in terms of J, H, K etc using $E = \frac{4\pi}{c} J$, $F = 4\pi H$, $P = \frac{4\pi}{c} K$

Then $\frac{4\pi}{c} J = \gamma^2 \left[\frac{4\pi}{c} J_0 + 2\beta \frac{4\pi}{c} H_0 + \beta^2 \frac{4\pi}{c} K_0 \right]$

$$(9.13') \quad \Rightarrow J = \gamma^2 [J_0 + 2\beta H_0 + \beta^2 K_0] \quad (9.13')$$

$$4\pi H = \gamma^2 [(1+\beta^2) 4\pi H_0 + \beta \frac{4\pi}{c} J_0 + \beta \frac{4\pi}{c} K_0]$$

$$(9.14') \quad H = \gamma^2 [(1+\beta^2) H_0 + \beta J_0 + \beta K_0] \quad (9.14')$$

$$\frac{4\pi}{c} K = \gamma^2 \left(\frac{4\pi}{c} K_0 + 2\beta \frac{4\pi}{c} H_0 + \beta^2 \frac{4\pi}{c} J_0 \right)$$

$$(9.15') \quad K = \gamma^2 (K_0 + 2\beta H_0 + \beta^2 J_0)$$

Then

$$c^{-1} \frac{\partial}{\partial t} \left\{ \gamma^2 [J_0 + 2\beta H_0 + \beta^2 K_0] \right\} + \frac{\partial}{\partial r} \left\{ \gamma^2 [(1+\beta^2) H_0 + \beta J_0 + \beta K_0] \right\}$$

$$+ \frac{2}{r} \left\{ \gamma^2 [(1+\beta^2) H_0 + \beta J_0 + \beta K_0] \right\} = - \frac{c}{4\pi} G^0$$

$$= c^{-1} \frac{\partial \gamma^2}{\partial t} [J_0 + 2\beta H_0 + \beta^2 K_0] + c^{-1} \gamma^2 \left[\frac{\partial J_0}{\partial t} + 2 \frac{\partial}{\partial t} (\beta H_0) + \frac{\partial}{\partial t} (\beta^2 K_0) \right]$$

$$+ \frac{\partial \gamma^2}{\partial r} [(1+\beta^2) H_0 + \beta J_0 + \beta K_0] + \gamma^2 \left\{ \frac{\partial}{\partial r} [(1+\beta^2) H_0] + \frac{\partial}{\partial r} (\beta J_0) + \frac{\partial}{\partial r} (\beta K_0) \right\}$$

$$+ \frac{2\gamma^2}{r} \left\{ (1+\beta^2) H_0 + \beta J_0 + \beta K_0 \right\} = - \frac{c}{4\pi} G^0$$

now evaluate $\frac{\partial \gamma^2}{\partial t} = 2\gamma \frac{\partial}{\partial t} (1-\beta^2)^{-1/2} = 2\gamma \frac{\partial \beta}{\partial t} (1-\beta^2)^{-3/2} = 2\gamma^3 \frac{\partial \beta}{\partial t}$

$$= 2\gamma \cdot \gamma^3 \cdot \beta \frac{\partial \beta}{\partial t} = 2\gamma^4 \beta \frac{\partial \beta}{\partial t}$$

Similarly $\frac{\partial \gamma^2}{\partial r} = 2\gamma^4 \beta \frac{\partial \beta}{\partial r}$

Hence we have:

$$\begin{aligned} & c^{-1} \left(2\gamma^4 \beta \frac{\partial \beta}{\partial t} \right) (J_0 + 2\beta H_0 + \beta^2 K_0) + c^{-1} \gamma^2 \left(\frac{\partial J_0}{\partial t} + 2\beta \frac{\partial H_0}{\partial t} + \beta^2 \frac{\partial K_0}{\partial t} \right) + c^{-1} \gamma^2 \frac{\partial \beta}{\partial t} (2H_0 + 2\beta K_0) \\ & + 2\gamma^4 \beta \frac{\partial \beta}{\partial r} \left[(1+\beta^2) H_0 + \beta J_0 + \beta K_0 \right] + \gamma^2 \left\{ (1+\beta^2) \frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} + \beta \frac{\partial K_0}{\partial r} \right\} + \gamma^2 \frac{\partial \beta}{\partial r} [2\beta H_0 + J_0 + K_0] \\ & + \frac{2\gamma^2}{r} \left\{ (1+\beta^2) H_0 + \beta J_0 + \beta K_0 \right\} = -\frac{c}{4\pi} G^0 \end{aligned}$$

$$\begin{aligned} \text{or } & c^{-1} \gamma^2 \left(\frac{\partial J_0}{\partial t} + 2\beta \frac{\partial H_0}{\partial t} + \beta^2 \frac{\partial K_0}{\partial t} \right) + \gamma^2 \left\{ (1+\beta^2) \frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} + \beta \frac{\partial K_0}{\partial r} \right\} \\ & + c^{-1} \gamma^2 \frac{\partial \beta}{\partial t} \left\{ 2\gamma^2 \beta (J_0 + 2\beta H_0 + \beta^2 K_0) + (2H_0 + 2\beta K_0) \right\} \\ & + \gamma^2 \frac{\partial \beta}{\partial r} \left\{ 2\gamma^2 \beta [(1+\beta^2) H_0 + \beta J_0 + \beta K_0] + (2\beta H_0 + J_0 + K_0) \right\} \\ & + \frac{2\gamma^2}{r} \left\{ (1+\beta^2) H_0 + \beta J_0 + \beta K_0 \right\} = -\frac{c}{4\pi} G^0 = -\frac{c}{4\pi} \gamma (G_0^0 + \beta G_0^1) \end{aligned}$$

$$\begin{aligned} \text{And } \frac{1}{4\pi} G_0^0 &= -c^{-1} \int dv_0 [\eta_0(v_0) - \chi_0(v_0) J_0(v_0)] \\ \frac{1}{4\pi} G_0^1 &= -c^{-1} \int dv_0 [-\chi_0(v_0) + H_0(v_0)] \end{aligned}$$

hence

$$\begin{aligned} \frac{c}{4\pi} G_0^0 &= - \int dv_0 [\eta_0(v_0) - \chi_0(v_0) J_0(v_0)] \\ \frac{c}{4\pi} G_0^1 &= + \int \chi_0(v_0) H_0(v_0) dv_0 \end{aligned}$$

$$-\frac{c}{4\pi} G^0 = +\gamma \int dv_0 [\eta_0(v_0) - \chi_0(v_0) J_0(v_0)] - \gamma\beta \int \chi_0(v_0) H_0(v_0) dv_0$$

We now have

$$\begin{aligned} & \frac{\gamma}{c} \left[\left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \beta \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) \right] + \gamma \left[\left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \beta \left(\frac{\partial K_0}{\partial r} + \beta \frac{\partial H_0}{\partial r} \right) \right] \\ & + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left\{ 2\beta J_0 + 4\beta^2 H_0 + 2\beta^3 K_0 + 2(1-\beta^2) H_0 + 2\beta(1-\beta^2) K_0 \right\} \\ & + \gamma^3 \frac{\partial \beta}{\partial r} \left\{ 2\beta(1+\beta^2) H_0 + \beta^2 J_0 + \beta^2 K_0 + 2\beta(1-\beta^2) H_0 + (1-\beta^2) J_0 + (1-\beta^2) K_0 \right\} \\ & + \frac{\gamma}{r} \left\{ 2H_0 + 2\beta^2 H_0 + 2\beta J_0 + 2\beta K_0 \right\} = \int dv_0 [\eta_0(v_0) - \chi_0(v_0) J_0(v_0)] - \beta \int \chi_0(v_0) H_0(v_0) dv_0 \end{aligned}$$

$$\begin{aligned}
& \frac{\gamma}{c} \left[\left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \beta \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) \right] + \gamma \left[\left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \beta \left(\frac{\partial K_0}{\partial r} + \beta \frac{\partial H_0}{\partial r} \right) \right] \\
& + \frac{\gamma}{r} \left[(2H_0 + 3\beta J_0 - \beta K_0) + (-\beta J_0 + 2\beta^2 H_0 + 3\beta K_0) \right] \\
& + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left[\underbrace{2\beta J_0 + 2\beta^2 H_0 + 2H_0 + 2\beta K_0}_{(2H_0 + \beta J_0 + \beta K_0) + 3(2\beta H_0 + J_0 + K_0)} \right] + \gamma^3 \frac{\partial \beta}{\partial r} \left[\underbrace{4\beta H_0 + J_0 + K_0 + \beta^2 J_0 + \beta^2 K_0}_{(J_0 + K_0 + 2\beta H_0) + \beta(2H_0 + \beta J_0 + \beta K_0)} \right] \\
& = \frac{\gamma}{c} \left[\left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) + \beta \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) \right] + \gamma \left[\left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) + \beta \left(\frac{\partial K_0}{\partial r} + \beta \frac{\partial H_0}{\partial r} \right) \right] \\
& + \frac{\gamma}{r} \left[(2H_0 + 3\beta J_0 - \beta K_0) + \beta (-J_0 + K_0 + 2\beta H_0) \right] + \frac{\gamma^3}{c} \frac{\partial \beta}{\partial t} \left[(2H_0 + \beta J_0 + \beta K_0) + \beta (H_0 + K_0 + 2\beta H_0) \right] \\
& + \gamma^3 \frac{\partial \beta}{\partial r} \left[(J_0 + K_0 + 2\beta H_0) + \beta (2H_0 + \beta J_0 + \beta K_0) \right] = \int dV_0 (\eta_0(v_0) - \chi_0(v_0) \mathcal{L}(v_0)) - \beta \int dV_0 \chi_0(v_0) H_0
\end{aligned}$$

In each bracket the 1st parentheses contains the appropriate term from eqn (95.13) and the second parentheses, which is multiplied by β , contains the term from (95.14).

Similarly for (95.16)

$$c^{-1} \partial H / \partial t + \partial K / \partial r + (3K - J) / r = -(c/4\pi) G'$$

becomes

$$\begin{aligned}
c^{-1} \frac{\partial}{\partial t} \left\{ \gamma^2 [(1 + \beta^2) H_0 + \beta J_0 + \beta K_0] \right\} + \frac{\partial}{\partial r} \left\{ \gamma^2 [K_0 + 2\beta H_0 + \beta^2 J_0] \right\} \\
+ \frac{1}{r} \left\{ 3\gamma^2 (K_0 + 2\beta H_0 + \beta^2 J_0) - \gamma^2 (J_0 + 2\beta H_0 + \beta^2 K_0) \right\} = -\frac{c}{4\pi} \gamma (G'_0 + \beta G''_0)
\end{aligned}$$

\Rightarrow

$$\begin{aligned}
\gamma c^{-1} \left\{ \frac{\partial H_0}{\partial t} + \beta^2 \frac{\partial H_0}{\partial t} + \beta \frac{\partial J_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right\} + \gamma c^{-1} \frac{\partial \beta}{\partial t} \left\{ 2\gamma^2 \beta [H_0 + \beta^2 H_0 + \beta J_0 + \beta K_0] + 2\beta H_0 + J_0 + K_0 \right\} \\
+ \gamma \left\{ \frac{\partial K_0}{\partial r} + 2\beta \frac{\partial H_0}{\partial r} + \beta^2 \frac{\partial J_0}{\partial r} \right\} + \gamma \frac{\partial \beta}{\partial r} \left\{ 2\gamma^2 \beta [K_0 + 2\beta H_0 + \beta^2 J_0] + 2H_0 + 2\beta J_0 \right\} \\
+ \frac{\gamma^2}{r} \left\{ (3 - \beta^2) K_0 + 4\beta H_0 + (3\beta^2 - 1) J_0 \right\} = -\int \chi_0 H_0 dV_0 + \beta \int (\eta_0 - \chi_0 J_0) dV_0
\end{aligned}$$

$$\begin{aligned}
\text{now } 2\gamma^2 \beta [H_0 + \beta^2 H_0 + \beta J_0 + \beta K_0] + 2\beta H_0 + J_0 + K_0 &= \gamma^2 \{ 2\beta H_0 + 2\beta^3 H_0 + 2\beta^2 J_0 + 2\beta^2 K_0 + 2(1 - \beta^2)\beta H_0 + (1 - \beta^2)J_0 + H_0 \} \\
&= \gamma^2 \{ 2\beta H_0 + 2\beta^3 H_0 + 2\beta^2 J_0 + 2\beta^2 K_0 + 2\beta H_0 - 2\beta^3 H_0 + J_0 + K_0 - \beta^2 J_0 - \beta^2 K_0 \} = \gamma^2 \{ J_0 + K_0 + 4\beta H_0 + \beta^2 J_0 + \beta^2 K_0 \}
\end{aligned}$$

$$\text{and } 2\gamma^2\beta(K_0 + 2\beta H_0 + \beta^2 J_0) + 2H_0 + 2\beta J_0 = \gamma^2 \{ 2\beta K_0 + 4\beta^2 H_0 + 2\beta^3 J_0 + (1-\beta^2)(2H_0 + 2\beta J_0) \}$$

$$= \gamma^2 \{ 2\beta K_0 + 4\beta^2 H_0 + 2\beta^3 J_0 + 2H_0 + 2\beta J_0 - 2\beta^2 H_0 - 2\beta^3 J_0 \} = \gamma^2 \{ 2H_0 + 2\beta J_0 + 2\beta K_0 + 2\beta^2 H_0 \}$$

Finally, we can also write these results:

$$\gamma^2 (J_0 + K_0 + 4\beta H_0 + \beta^2 J_0 + \beta^2 K_0) = \gamma^2 \{ (J_0 + K_0 + 2\beta H_0) + \beta(2H_0 + \beta J_0 + \beta K_0) \}$$

$$\text{and } \gamma^2 (2H_0 + 2\beta J_0 + 2\beta K_0 + 2\beta^2 H_0) = \gamma^2 \{ (2H_0 + \beta J_0 + \beta K_0) + \beta(J_0 + K_0 + 2\beta H_0) \}$$

We also find

$$\frac{\gamma^2}{\gamma} \{ (3-\beta^2)K_0 + 4\beta H_0 + (3\beta^2-1)J_0 \} = \frac{\gamma}{\gamma} \{ 3K_0 - J_0 + 4\beta H_0 - \beta^2 K_0 + 3\beta^2 J_0 \}$$

$$= \frac{\gamma}{\gamma} \{ (3K_0 - J_0 + 2\beta H_0) + \beta(2H_0 + 3\beta J_0 - \beta K_0) \}$$

Hence the entire equation becomes:

$$c^{-1} \left\{ \left(\frac{\partial H_0}{\partial t} + \beta \frac{\partial K_0}{\partial t} \right) + \beta \left(\frac{\partial J_0}{\partial t} + \beta \frac{\partial H_0}{\partial t} \right) \right\} + c^{-1} \gamma^3 \frac{\partial \beta}{\partial t} \left\{ (J_0 + K_0 + 2\beta H_0) + \beta(2H_0 + \beta J_0 + \beta K_0) \right\}$$

$$+ \left\{ \left(\frac{\partial K_0}{\partial r} + \beta \frac{\partial H_0}{\partial r} \right) + \beta \left(\frac{\partial H_0}{\partial r} + \beta \frac{\partial J_0}{\partial r} \right) \right\} + \gamma^3 \frac{\partial \beta}{\partial r} \left\{ (2H_0 + \beta J_0 + \beta K_0) + \beta(J_0 + K_0 + 2\beta H_0) \right\}$$

$$+ \frac{\gamma}{\gamma} \left\{ (3K_0 - J_0 + 2\beta H_0) + \beta(2H_0 + 3\beta J_0 - \beta K_0) \right\} = - \int \lambda_0 H_0 dv_0 + \int (\eta_0 - \lambda_0 J_0) dv_0$$

which is exactly (95.14) + β times (95.13),

Equation (95.9) to 1st order in (v/c) becomes:

$$\frac{1}{c} \frac{\partial I_0}{\partial t} + (\mu_0 + \beta) \frac{\partial I_0}{\partial r} + (1-\mu_0^2) \left\{ r' (1+\beta\mu_0) - c^{-1} \frac{\partial \beta}{\partial t} - (\mu_0 \frac{\partial \beta}{\partial r}) \right\} \frac{\partial I_0}{\partial \mu_0}$$

$$- \left\{ r' (1-\mu_0^2) \beta + c^{-1} \mu_0 \frac{\partial \beta}{\partial t} + \mu_0^2 \frac{\partial \beta}{\partial r} \right\} \nu_0 \frac{\partial I_0}{\partial \nu_0} + 3 \left\{ r' (1-\mu_0^2) \beta + \mu_0 c^{-1} \frac{\partial \beta}{\partial t} + \mu_0^2 \frac{\partial \beta}{\partial r} \right\} I_0$$

$$= \eta_0 - \lambda_0 I_0$$

or

$$c^{-1} \left\{ \frac{\partial I_0}{\partial t} + v \frac{\partial I_0}{\partial r} \right\} + \mu_0 \frac{\partial I_0}{\partial r} + (1-\mu_0^2) \left\{ \frac{1}{r} + (\mu_0/c) [v/r - \partial v/\partial r] - a/c^2 \right\} \frac{\partial I_0}{\partial \mu_0}$$

$$- c^{-1} \left\{ (1-\mu_0^2) v/r + \mu_0^2 \partial v/\partial r + \mu_0 a/c^2 \right\} \nu_0 \frac{\partial I_0}{\partial \nu_0} + 3 \left\{ (1-\mu_0^2) v/r + \mu_0^2 \partial v/\partial r + \mu_0 a/c^2 \right\} I_0$$

$$= \eta_0 - \lambda_0 I_0$$

$$\Rightarrow \frac{1}{c} \frac{DI_0}{Dt} + \mu_0 \frac{\partial I_0}{\partial r} + (1-\mu_0^2) \left\{ \frac{1}{r} + \frac{\mu_0}{c} \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} - \frac{1}{c} \left\{ (1-\mu_0^2) \frac{v}{r} + \mu_0^2 \frac{\partial v}{\partial r} + \frac{\mu_0 a}{c} \right\} \nu_0 \frac{\partial I_0}{\partial \nu_0}$$

$$(95.17) \quad + \frac{3}{c} \left\{ (1-\mu_0^2) \frac{v}{r} + \mu_0^2 \frac{\partial v}{\partial r} + \frac{\mu_0 a}{c} \right\} I_0 = \eta_0 - \lambda_0 I_0$$

which is (95.17) written slightly differently than in the text.

Equation (95.10) to 1st order in v/c is

$$c' \left\{ \frac{\partial J_0}{\partial t} + \frac{v}{c} \frac{\partial H_0}{\partial t} \right\} + \left\{ \frac{\partial H_0}{\partial r} + \frac{v}{c} \frac{\partial J_0}{\partial r} \right\} - \gamma_0 \left\{ \frac{v}{rc} \left[\frac{\partial J_0}{\partial v_0} - \frac{\partial K_0}{\partial v_0} \right] + \frac{a}{c^2} \left[\frac{\partial H_0}{\partial v_0} \right] + \frac{1}{c} \frac{\partial v}{\partial r} \frac{\partial K_0}{\partial v_0} \right\} \\ + \left\{ \frac{2}{r} \left[H_0 + \frac{v}{c} J_0 \right] + \frac{a}{c^2} H_0 + \frac{1}{c} \frac{\partial v}{\partial r} J_0 \right\} = \eta_0 - \chi_0 \bar{J}_0$$

or

$$\frac{1}{c} \frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial r} - \frac{\gamma_0}{c} \left\{ \frac{v}{r} \frac{\partial J_0}{\partial v_0} - \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) \frac{\partial K_0}{\partial v_0} + \frac{a}{c} \frac{\partial H_0}{\partial v_0} \right\} + \left\{ \left(\frac{2v}{r} + \frac{\partial v}{\partial r} \right) J_0 + \left(\frac{2}{r} + \frac{a}{c^2} \right) H_0 \right\}$$

$$= \frac{1}{c} \frac{\partial J_0}{\partial t} + \frac{\partial H_0}{\partial r} + \frac{2H_0}{r} + \left(\frac{2v}{r} + \frac{\partial v}{\partial r} \right) \frac{J_0}{c} - \frac{\gamma_0}{c} \left\{ \frac{v}{r} \frac{\partial J_0}{\partial v_0} - \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) \frac{\partial K_0}{\partial v_0} + \frac{a}{c} \frac{\partial H_0}{\partial v_0} \right\} = \eta_0 - \chi_0 J_0 \\ + \frac{a}{c^2} H_0$$

(95.18)

$$= \frac{1}{c} \frac{\partial J_0}{\partial t} + \frac{\partial H_0}{\partial r} + \frac{2H_0}{r} + \left(\frac{2v}{r} + \frac{\partial v}{\partial r} \right) \frac{J_0}{c} + \frac{a}{c^2} H_0 - \frac{\gamma_0}{c} \left\{ \frac{v}{r} \frac{\partial J_0}{\partial v_0} - \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) \frac{\partial K_0}{\partial v_0} + \frac{a}{c} \frac{\partial H_0}{\partial v_0} \right\} = \eta_0 - \chi_0 J_0$$

And (95.12) becomes

$$\frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{\partial K_0}{\partial r} + \frac{v}{c} \frac{\partial H_0}{\partial r} - \frac{\gamma_0}{c} \left\{ \frac{v}{r} \left(\frac{\partial H_0}{\partial v_0} - \frac{\partial N_0}{\partial v_0} \right) + \frac{a}{c} \frac{\partial K_0}{\partial v_0} + \frac{\partial v}{\partial r} \frac{\partial N_0}{\partial v_0} \right\} + \left\{ \frac{1}{r} (3K_0 - J_0) + \frac{v}{cr} (H_0 + N_0) \right\} \\ + \frac{a}{c^2} \left[J_0 \right] + \frac{1}{c} \frac{\partial v}{\partial r} (2H_0 - N_0) = -\chi_0 H_0$$

$$\frac{1}{c} \left(\frac{\partial H_0}{\partial t} + v \frac{\partial H_0}{\partial r} \right) + \frac{\partial K_0}{\partial r} + \frac{1}{r} (3K_0 - J_0) + \frac{1}{c} \left(\frac{v}{r} + \frac{\partial v}{\partial r} \right) H_0 + \frac{1}{c} \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) N_0 + \frac{a}{c^2} J_0$$

$$- \frac{\gamma_0}{c} \left\{ \frac{v}{r} \frac{\partial H_0}{\partial v_0} - \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) \frac{\partial N_0}{\partial v_0} + \frac{a}{c} \frac{\partial K_0}{\partial v_0} \right\} = -\chi_0 H_0$$

$$\frac{1}{c} \frac{DH_0}{Dt} + \frac{\partial K_0}{\partial r} + \frac{1}{r} (3K_0 - J_0) + \frac{1}{c} \left(\frac{v}{r} + \frac{\partial v}{\partial r} \right) H_0 + \frac{1}{c} \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) N_0 + \frac{a}{c^2} J_0$$

(95.19)

$$- \frac{\gamma_0}{c} \left\{ \frac{v}{r} \frac{\partial H_0}{\partial v_0} - \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) \frac{\partial N_0}{\partial v_0} + \frac{a}{c} \frac{\partial K_0}{\partial v_0} \right\} = -\chi_0 H_0$$

Further, the integrated moment eqns become (95.13):

$$\frac{1}{c} \left(\frac{\partial J_0}{\partial t} + \frac{v}{c} \frac{\partial H_0}{\partial t} \right) + \frac{\partial H_0}{\partial r} + \frac{v}{c} \frac{\partial J_0}{\partial r} + \left\{ \frac{1}{r} \left(2H_0 + 3 \frac{v}{c} J_0 - \frac{v}{c} K_0 \right) + \frac{a}{c^2} (2H_0) + \frac{1}{c} \frac{\partial v}{\partial r} (J_0 + K_0) \right\} \\ = \int (\eta_0 - \chi_0 J_0) dv_0$$

or

$$\frac{1}{c} \frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial r} + \frac{2H_0}{r} + \frac{1}{c} \left(\frac{3v}{r} + \frac{\partial v}{\partial r} \right) J_0 + \frac{1}{c} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) K_0 + \frac{2a}{c^2} H_0 = \int (\eta_0 - \chi_0 J_0) dv_0$$

(95.20)

$$\frac{1}{c} \frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial r} + \frac{2H_0}{r} + \frac{\partial v}{cr} (3J_0 - K_0) + \frac{1}{c} \frac{\partial v}{\partial r} (J_0 + K_0) + \frac{2a}{c^2} H_0 = \int dv_0 (\eta_0 - \chi_0 J_0)$$

And (15.14) becomes:

$$\frac{1}{c} \left(\frac{\partial H_0}{\partial t} \right) + \left(\frac{\partial K_0}{\partial r} + \frac{v}{c} \frac{\partial H_0}{\partial r} \right) + \frac{1}{r} (3K_0 - J_0 + \frac{2v}{c} H_0) + \frac{a}{c^2} (J_0 + K_0) + \frac{1}{c} \frac{\partial v}{\partial r} (2H_0) = - \int x_0 H_0 dv_0$$

$$\Rightarrow \frac{1}{c} \frac{DH_0}{Dt} + \frac{\partial K_0}{\partial r} + \frac{2v}{c} \frac{H_0}{r} + \frac{1}{r} (3K_0 - J_0) + \frac{2 \partial v}{c \partial r} H_0 + \frac{a}{c^2} (J_0 + K_0) = - \int x_0 H_0 dv_0$$

$$\text{or} \\ (95.21) \quad \frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{\partial K_0}{\partial r} + \left(\frac{v}{r} + \frac{\partial v}{\partial r} \right) \frac{2H_0}{c} + \frac{1}{r} (3K_0 - J_0) + \frac{a}{c^2} (J_0 + K_0) = - \int x_0 H_0 dv_0$$

Consistency of inertial frame & co-moving frame equals to $O(v/c)$ in grey, LTE, nonscattering planar medium; with no acceleration terms:

$$(95.20) \rightarrow \frac{1}{c} \left(\frac{\partial J_0}{\partial t} + v \frac{\partial J_0}{\partial z} \right) + \frac{\partial H_0}{\partial z} + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) = \int dv_0 (\eta_0 - x_0 J_0) = x_0 (B_0 - J_0)$$

Now take (93.10)

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial z} = x (4\pi B - cE) + \frac{x}{c} VF \Rightarrow \frac{4\pi}{c} \frac{\partial J}{\partial t} + 4\pi \frac{\partial H}{\partial z} = x_0 (4\pi B_0 - c \frac{4\pi}{c} J) - \frac{4\pi v}{c} 4\pi H$$

$$\text{or} \quad \frac{1}{c} \frac{\partial J}{\partial t} + \frac{\partial H}{\partial z} = x_0 (B_0 - J) + \frac{xv}{c} H$$

$$\Rightarrow \frac{1}{c} \frac{\partial}{\partial t} [J_0 + 2\beta H_0] + \frac{\partial}{\partial z} [H_0 + \beta (J_0 + K_0)] = x_0 [B_0 - (J_0 + 2\beta H_0)] + \frac{x_0 v}{c} [H_0 + \beta (J_0 + K_0)]$$

$$= \frac{1}{c} \frac{\partial J_0}{\partial t} + \frac{2v}{c^2} \frac{\partial H_0}{\partial t} + \frac{2a}{c^2} H_0 + \frac{\partial H_0}{\partial z} + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) + \beta \left(\frac{\partial J_0}{\partial z} + \frac{\partial K_0}{\partial z} \right) = x_0 (B_0 - J_0 - 2\beta H_0) + (x_0 v/c) H_0$$

$$\frac{1}{c} \left(\frac{\partial J_0}{\partial t} + v \frac{\partial J_0}{\partial z} \right) + \frac{\partial H_0}{\partial z} + \frac{2a}{c^2} H_0 + \frac{v}{c} \left(\frac{\partial J_0}{\partial z} + \frac{\partial K_0}{\partial z} \right) + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) = x_0 (B_0 - J_0 - \beta H_0)$$

$$\Rightarrow \frac{1}{c} \frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial z} + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) + \frac{v}{c} \left(\frac{\partial J_0}{\partial z} + \frac{\partial K_0}{\partial z} \right) + x_0 \beta H_0 = x_0 (B_0 - J_0)$$

$$\text{or} \quad \frac{1}{c} \frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial z} + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) = x_0 (B_0 - J_0) - \left[x_0 \beta H_0 + \frac{v}{c} \left(\frac{\partial J_0}{\partial z} + \frac{\partial K_0}{\partial z} \right) \right] \\ = x_0 (B_0 - J_0) - \beta [x_0 H_0 + \partial K_0 / \partial z]$$

But from (95.21)

$$-x_0 H_0 - \frac{\partial K_0}{\partial z} \approx \frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{v}{c} \left\{ \frac{\partial H_0}{\partial t} \right\} + \frac{2}{c} \frac{\partial v}{\partial z} H_0 = O(v/c)$$

Hence $\frac{v}{c} [\alpha_0 H_0 + \partial k_0 / \partial z] = \frac{v}{c} \times \alpha (\frac{v}{c}) = \alpha (v^2/c^2)$

and

$$\frac{1}{c} \frac{\partial J_0}{\partial t} + \frac{\partial H_0}{\partial z} + \frac{1}{c} \frac{\partial v}{\partial z} (J_0 + K_0) = \alpha_0 (B_0 - J_0) - \alpha (v^2/c^2)$$

Conversely, starting with (95, 21):

$$\frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{\partial K_0}{\partial z} + \frac{v}{c} \frac{\partial H_0}{\partial z} + \frac{2}{c} \frac{\partial v}{\partial z} H_0 = -\alpha_0 H_0$$

and (93.11)

$$c^2 \partial F / \partial t + \partial P / \partial z = -(\alpha_0/c) F + (\alpha_0 v/c) (4\pi B_0) + \alpha_0 v P$$

$$\Rightarrow c^2 \frac{\partial}{\partial t} (4\pi H) + \frac{\partial}{\partial z} (4\pi K) = -\frac{\alpha_0}{c} (4\pi H) + \frac{\alpha_0 v}{c} (4\pi B) + \alpha_0 v \frac{\partial}{\partial z} (4\pi K)$$

$$\Rightarrow \frac{1}{c} \frac{\partial H}{\partial t} + \frac{\partial K}{\partial z} = -\alpha_0 H + \frac{\alpha_0 v}{c} B + \frac{\alpha_0 v K}{c} = -\alpha_0 H + (\alpha_0 v/c) B_0 + (\alpha_0 v/c) K$$

and thus

$$\frac{1}{c} \frac{\partial}{\partial t} [H_0 + \beta (J_0 + K_0)] + \frac{\partial}{\partial z} [K_0 + 2\beta H_0] = -\alpha_0 [H_0 + \beta (J_0 + K_0)] + \frac{\alpha_0 v}{c} (B_0 + K_0 + 2\beta H_0)$$

$$\left(\frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{v}{c} \frac{\partial H_0}{\partial z} \right) + \frac{\partial K_0}{\partial z} + \frac{v}{c} \frac{\partial H_0}{\partial z} + \frac{2}{c} \frac{\partial v}{\partial z} H_0 = -\alpha_0 H_0 + \alpha_0 \frac{v}{c} (J_0 + K_0) + \alpha_0 \frac{v}{c} (B_0 + K_0) \\ = -\alpha_0 H_0 + \alpha_0 \frac{v}{c} (B_0 - J_0)$$

$$\frac{1}{c} \frac{\partial H_0}{\partial t} + \frac{\partial K_0}{\partial z} + \frac{v}{c} \frac{\partial H_0}{\partial z} + \frac{2}{c} \frac{\partial v}{\partial z} H_0 = -\alpha_0 H_0 + \alpha_0 \frac{v}{c} (B_0 - J_0) - \frac{v}{c} \frac{\partial H_0}{\partial z}$$

Noninertial Frame Formulation (LH Thomas + J. Castor)

Start with photon Boltzmann eqn (§92)

$$(95.30) \quad m^\alpha \frac{\partial \mathcal{J}}{\partial x^\alpha} + \dot{m}^\alpha \frac{\partial \mathcal{J}}{\partial m^\alpha} = e - a \mathcal{J} = \frac{\delta \mathcal{J}}{\delta \lambda} \Big|_{\text{cell}}$$

and take

$$\dot{m}^\alpha = \frac{dm^\alpha}{d\lambda} \quad \text{where } \lambda \text{ satisfies } p^\alpha (=m^\alpha) = \frac{dx^\alpha}{d\lambda}$$

The intrinsic derivative $\frac{\delta m^\alpha}{\delta \lambda} \equiv 0$ along photon trajectory

$$\frac{\delta m^\alpha}{\delta \lambda} = \frac{dm^\alpha}{d\lambda} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} m^\beta \frac{dx^\gamma}{d\lambda} = 0$$

Now change indices and write:

$$\frac{\delta m^\beta}{\delta \lambda} = \frac{dm^\beta}{d\lambda} + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} m^\alpha \frac{dx^\gamma}{d\lambda} = \frac{\partial m^\beta}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} m^\alpha \frac{dx^\gamma}{d\lambda}$$

now $(dx^\alpha/d\lambda) = m^\alpha$ and $m^\beta = \epsilon_c^\beta m^c \Rightarrow$

$$\frac{\delta m^\beta}{\delta \lambda} = m^\alpha \left[\frac{\partial}{\partial x^\alpha} (\epsilon_c^\beta m^c) + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} m^\gamma \right] = m^\alpha \left[\epsilon_{c,\alpha}^\beta m^c + \epsilon_c^\beta \frac{\partial m^c}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} m^\gamma \right]$$

$$= m^\alpha m^c \left[\epsilon_{c,\alpha}^\beta + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} \epsilon_c^\gamma \right] + \epsilon_c^\beta m^\alpha \frac{\partial m^c}{\partial x^\alpha}$$

$$= m^a m^c \epsilon_a^\alpha \left[\epsilon_{c,\alpha}^\beta + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} \epsilon_c^\gamma \right] + \epsilon_c^\beta \frac{dm^c}{d\lambda} = 0$$

now solve for last term:

$$\epsilon_c^\beta \frac{dm^c}{d\lambda} = -m^a m^c \epsilon_a^\alpha \left[\epsilon_{c,\alpha}^\beta + \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\} \epsilon_c^\gamma \right] = -m^a m^c \epsilon_a^\alpha \epsilon_{c;\alpha}^\beta$$

multiply by $\frac{d\lambda}{d\lambda} \epsilon_\beta^b$ to get

$$\epsilon_\beta^b \epsilon_c^\beta \frac{dm^c}{d\lambda} = \delta_c^b \frac{dm^c}{d\lambda} = \frac{dm^b}{d\lambda} = -m^a m^c (\epsilon_\beta^b \epsilon_a^\alpha \epsilon_{c;\alpha}^\beta)$$

Define the Pfaffian derivative $\partial_a \equiv \epsilon_a^\alpha (\partial/\partial x^\alpha)$

and Ricci rotation coefficient $\Gamma_{ac}^b = \epsilon_a^\alpha \epsilon_\beta^b \epsilon_{c;\alpha}^\beta$

The transfer eqn becomes:

~~$$m^a \frac{\partial \mathcal{J}}{\partial x^a} = m^a \epsilon_a^\alpha \frac{\partial \mathcal{J}}{\partial x^\alpha} = m^a \frac{\partial \mathcal{J}}{\partial x^a} = m^a \frac{\partial \mathcal{J}}{\partial x^a} = m^a \frac{\partial \mathcal{J}}{\partial x^a}$$~~

$$m^a \frac{D}{Dx^a} [\mathcal{J}(x^\alpha, m^a)] = \frac{\delta \mathcal{J}}{\delta \lambda} \Big|_{\text{cell}} = \frac{\partial \mathcal{J}}{\partial x^\alpha} \frac{dx^\alpha}{d\lambda} + \frac{\partial \mathcal{J}}{\partial m^b} \frac{dm^b}{d\lambda}$$

$$= m^a \epsilon_a^\alpha \frac{\partial \mathcal{J}}{\partial x^\alpha} + \frac{\partial \mathcal{J}}{\partial m^b} \frac{dm^b}{d\lambda} = m^a \partial_a \mathcal{J} + \frac{\partial \mathcal{J}}{\partial m^b} (-m^a m^c \Gamma_{ac}^b)$$

$$= m^a \left[\partial_a \mathcal{J} - m^c \Gamma_{ac}^b \frac{\partial \mathcal{J}}{\partial m^b} \right]$$

(95.44)

$$= m^a \left[\partial_a - m^c \Gamma_{ac}^b \frac{\partial}{\partial m^b} \right] \mathcal{J} = e - a \mathcal{J}$$

$$ds^2 = e^{-2\Phi} dt^2 + e^{2\Lambda} dr^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

$$g_{00} = -\exp(+2\Phi) \quad g_{11} = \exp(2\Lambda) \quad g_{22} = R^2 \quad g_{33} = R^2 \sin^2\theta$$

Non-zero elements

$$\left\{ \begin{matrix} i \\ i \end{matrix} \right\} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^i} \quad \left\{ \begin{matrix} i \\ j \end{matrix} \right\} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} \quad \left\{ \begin{matrix} i \\ j \end{matrix} \right\} = -\frac{1}{2g_{ii}} \frac{\partial g_{jj}}{\partial x^i}$$

Thus

95.46)

$$\left\{ \begin{matrix} 0 \\ 00 \end{matrix} \right\} = \frac{1}{2g_{00}} \frac{\partial g_{00}}{\partial t} = \frac{1}{2g_{00}} \left(+2 \frac{\partial \Phi}{\partial t} \right) g_{00} = + \frac{\partial \Phi}{\partial t}$$

$$\left\{ \begin{matrix} 0 \\ 01 \end{matrix} \right\} = \frac{1}{2g_{00}} \frac{\partial g_{00}}{\partial r} = \frac{1}{2g_{00}} \left(2 \frac{\partial \Phi}{\partial r} \right) g_{00} = \frac{\partial \Phi}{\partial r}$$

$$\left\{ \begin{matrix} 0 \\ 11 \end{matrix} \right\} = -\frac{1}{2g_{00}} \frac{\partial g_{11}}{\partial t} = -\frac{1}{2[-\exp(2\Phi)]} \cdot 2 \frac{\partial \Lambda}{\partial t} e^{2\Lambda} = \exp[2(\Lambda - \Phi)] \frac{\partial \Lambda}{\partial t}$$

$$\left\{ \begin{matrix} 1 \\ 00 \end{matrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{00}}{\partial r} = -\frac{1}{2\exp(2\Lambda)} \left(+2 \frac{\partial \Phi}{\partial r} \right) \exp(+2\Phi) = +\exp[-2(\Lambda + \Phi)] \frac{\partial \Phi}{\partial r}$$

$$\left\{ \begin{matrix} 1 \\ 10 \end{matrix} \right\} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial t} = \frac{1}{2g_{11}} \left(2 \frac{\partial \Lambda}{\partial t} \right) g_{11} = \frac{\partial \Lambda}{\partial t}$$

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} = \frac{1}{2g_{11}} \frac{\partial g_{11}}{\partial r} = \frac{1}{2g_{11}} \left(2 \frac{\partial \Lambda}{\partial r} \right) g_{11} = \frac{\partial \Lambda}{\partial r}$$

$$\left\{ \begin{matrix} 0 \\ 22 \end{matrix} \right\} = -\frac{1}{2g_{00}} \frac{\partial g_{22}}{\partial t} = +\frac{\exp(+2\Phi)}{2} 2R \frac{\partial R}{\partial t} = +\exp(+2\Phi) R \frac{\partial R}{\partial t} = +\exp(-2\Phi) R \frac{\partial R}{\partial t}$$

$$\left\{ \begin{matrix} 2 \\ 02 \end{matrix} \right\} = \frac{1}{2g_{22}} \frac{\partial g_{22}}{\partial t} = \frac{1}{2R^2} 2R \frac{\partial R}{\partial t} = \frac{1}{R} \frac{\partial R}{\partial t}$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial r} = -\frac{1}{2\exp(2\Lambda)} \cdot 2R \frac{\partial R}{\partial r} = -\exp(-2\Lambda) R \frac{\partial R}{\partial r}$$

$$\left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} = (2g_{22})^{-1} \partial g_{22} / \partial r = (2R^2)^{-1} \partial (R^2) / \partial r = R^{-1} \partial R / \partial r$$

$$\left\{ \begin{matrix} 0 \\ 33 \end{matrix} \right\} = -\frac{1}{2g_{00}} \frac{\partial g_{33}}{\partial t} = -\frac{1}{2g_{00}} \sin^2\theta \cdot 2R \frac{\partial R}{\partial t} = +\exp(-2\Phi) \sin^2\theta R \frac{\partial R}{\partial t}$$

$$\left\{ \begin{matrix} 3 \\ 03 \end{matrix} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial t} = \frac{1}{2R^2 \sin^2\theta} \cdot \sin^2\theta \cdot 2R \frac{\partial R}{\partial t} = \frac{1}{R} \frac{\partial R}{\partial t}$$

$$\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{33}}{\partial r} = -\frac{1}{2 \exp(2\Lambda)} \frac{\partial (R^2 \sin^2 \theta)}{\partial r} = -\exp(-2\Lambda) \sin^2 \theta R \frac{\partial R}{\partial r}$$

$$\left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial r} = \frac{1}{2R^2 \sin^2 \theta} \frac{\partial (R^2 \sin^2 \theta)}{\partial r} = \frac{1}{R} \frac{\partial R}{\partial r}$$

$$\left\{ \begin{matrix} 2 \\ 33 \end{matrix} \right\} = -\frac{1}{2g_{22}} \frac{\partial g_{33}}{\partial \theta} = -\frac{1}{2R^2} \frac{\partial (R^2 \sin^2 \theta)}{\partial \theta} = -\sin \theta \cos \theta$$

$$\left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial \theta} = \frac{1}{2R^2 \sin^2 \theta} \cdot R^2 \frac{\partial (\sin^2 \theta)}{\partial \theta} = \frac{\cos \theta}{\sin \theta} = \cot \theta$$

(95.46)



Orthonormal basis:

$$\underline{e}_0 = e^{-\Phi} \underline{e}_t \quad \underline{e}_1 = e^{-\Lambda} \underline{e}_r \quad \underline{e}_2 = R^{-1} \underline{e}_\theta \quad \underline{e}_3 = (R \sin \theta)^{-1} \underline{e}_\phi$$

or write

$$\underline{e}_a = e_a^\alpha \underline{e}_\alpha \quad \text{where } \underline{e}_a = (\underline{e}_0, \underline{e}_1, \underline{e}_2, \underline{e}_3)$$

$$\underline{e}_\alpha = (e_t, e_r, e_\theta, e_\phi)$$

Then

(95.48)

(95.49)

$$e_a^\alpha = \begin{pmatrix} e^\Phi & & & 0 \\ & e^\Lambda & & \\ 0 & & R^{-1} & \\ & & & (R \sin \theta)^{-1} \end{pmatrix} \quad \text{and} \quad e_\alpha^a = \begin{pmatrix} e^\Phi & & & 0 \\ & e^\Lambda & & \\ 0 & & R & \\ & & & R \sin \theta \end{pmatrix}$$

Photon 4-momentum
 m^0

Ricci rotation coefficients

For diagonal transformation matrix e^a , we find:

$$\Gamma_{ac}^b \equiv e^a_{\alpha} e^b_{\beta} e^{\gamma}_{c;\alpha} \equiv e^a_{\alpha} e^b_{\beta} [e^{\beta}_{c,\alpha} + \{\alpha\gamma\} e^{\gamma}_c]$$

becomes

$$\Gamma_{ac}^b = e^a_a (e^b_b)^{-1} [e^b_{c,a} \delta^b_c + \{ac\} e^c_c]$$

Now use (95.48) (95.4a) and (95.46) to evaluate the Γ_{ac}^b

$$\Gamma_{00}^1 = e^0_0 (e^1_1)^{-1} [e^1_{0,0} \delta^1_0 + \{00\} e^0_0] = e^{-\Phi} e^{-\Lambda} [e^{2(\Phi-\Lambda)} \frac{\partial \Phi}{\partial r} e^{-\Phi}] = e^{-\Lambda} \frac{\partial \Phi}{\partial r} = D_t \Phi$$

$$\Gamma_{22}^1 = e^2_2 (e^1_1)^{-1} [e^1_{2,2} \delta^1_2 + \{22\} e^2_2] = R^{-1} e^{-\Lambda} [-e^{-2\Lambda} R \frac{\partial R}{\partial r} R^{-1}] = -R^{-1} e^{-\Lambda} \frac{\partial R}{\partial r} = -R^{-1} \Gamma$$

$$\Gamma_{33}^1 = e^3_3 (e^1_1)^{-1} [e^1_{3,3} \delta^1_3 + \{33\} e^3_3] = (R \sin \theta)^{-1} e^{-\Lambda} [-e^{-2\Lambda} R \frac{\partial R}{\partial r} \sin \theta (R \sin \theta)^{-1}] = -R^{-1} e^{-\Lambda} \frac{\partial R}{\partial r} = -R^{-1} D_r R = -R^{-1} \Gamma$$

$$\Gamma_{10}^1 = e^1_1 (e^1_1)^{-1} [0 + \{10\} e^0_0] = 1 \left[\frac{\partial \Lambda}{\partial t} e^{-\Phi} \right] = D_t \Lambda$$

$$\Gamma_{33}^2 = e^3_3 (e^1_1)^{-2} [0 + \{33\} e^3_3] = (R \sin \theta)^2 R (-\sin \theta \cos \theta) = -R^{-1} \cot \theta$$

$$\Gamma_{20}^2 = e^2_2 (e^1_1)^{-2} [0 + \{20\} e^0_0] = 1 \left[R^{-1} \frac{\partial R}{\partial t} e^{-\Phi} \right] = R^{-1} e^{-\Phi} \frac{\partial R}{\partial t} = R^{-1} D_t R = R^{-1} U$$

$$\Gamma_{30}^3 = e^3_3 (e^1_1)^{-3} [0 + \{30\} e^0_0] = 1 \left[R^{-1} \frac{\partial R}{\partial t} e^{-\Phi} \right] = R^{-1} U$$

$$\Gamma_{21}^2 = e^2_2 (e^1_1)^{-2} [0 + \{21\} e^1_1] = 1 \left[R^{-1} \frac{\partial R}{\partial r} e^{-\Lambda} \right] = R^{-1} e^{-\Lambda} \frac{\partial R}{\partial r} = R^{-1} D_r R = R^{-1} \Gamma$$

$$\Gamma_{31}^3 = e^3_3 (e^1_1)^{-3} [0 + \{31\} e^1_1] = 1 \left[R^{-1} \frac{\partial R}{\partial r} e^{-\Lambda} \right] = R^{-1} \Gamma$$

$$\Gamma_{32}^3 = e^3_3 (e^1_1)^{-3} [0 + \{32\} e^2_2] = 1 \left[\cot \theta (R \sin \theta) \right] = R^{-1} \cot \theta$$

$$\Gamma_{11}^1 = e^1_1 (e^1_1)^{-1} [e^1_{1,1} + \{11\} e^1_1] = 1 \left[\frac{\partial}{\partial r} (e^{-\Lambda}) + \frac{\partial \Lambda}{\partial r} e^{-\Lambda} \right] = -\frac{\partial \Lambda}{\partial r} e^{-\Lambda} + \frac{\partial \Lambda}{\partial r} e^{-\Lambda} = 0$$

$$\frac{\partial}{\partial m^1} = \cos\theta \frac{\partial}{\partial v} - \frac{\sin\theta}{v} \frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial v} - \frac{\sin\theta}{v} \frac{(-\sin\theta) \frac{\partial}{\partial \theta}}{(-\sin\theta) \partial \theta} = \cos\theta \frac{\partial}{\partial v} + \frac{\sin^2\theta}{v} \frac{\partial}{\partial \mu}$$

$$= \mu \frac{\partial}{\partial v} + \frac{(1-\mu^2)}{v} \frac{\partial}{\partial \mu}$$

$$\frac{\partial}{\partial m^2} = \sin\theta \cos\Phi \frac{\partial}{\partial v} + v^{-1} \cos\theta \cos\Phi \frac{\partial}{\partial \theta} = \sin\theta \cos\Phi \frac{\partial}{\partial v} + \frac{\cos\theta \cos\Phi (-\sin\theta) \frac{\partial}{\partial \theta}}{(-\sin\theta) \partial \theta}$$

$$= \cos\Phi \left[\sin\theta \frac{\partial}{\partial v} - \frac{\cos\theta \sin\theta}{v} \frac{\partial}{\partial \mu} \right] = (1-\mu^2)^{1/2} \cos\Phi \left(\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right)$$

$$\frac{\partial}{\partial m^3} = \sin\theta \sin\Phi \frac{\partial}{\partial v} + \frac{\cos\theta \sin\Phi}{v} \frac{\partial}{\partial \theta} = \sin\Phi \left[\sin\theta \frac{\partial}{\partial v} + \frac{\cos\theta (-\sin\theta)}{v} \frac{\partial}{\partial \theta} \right]$$

$$= (1-\mu^2)^{1/2} \sin\Phi \left[\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right]$$

$$m^a \partial_a = m^a e_a^\alpha \frac{\partial}{\partial x^\alpha} = m^0 e_0^\alpha \frac{\partial}{\partial t} + m^1 e_1^\alpha \frac{\partial}{\partial r}$$

$$= v \bar{e}^\Phi \frac{\partial}{\partial t} + v \cos \Phi \bar{e}^\Lambda \frac{\partial}{\partial r} = v \bar{e}^\Phi \frac{\partial}{\partial t} + v \mu \bar{e}^\Lambda \frac{\partial}{\partial r} \equiv v D_t + \mu v D_r$$

$$m^a m^c \Gamma_{ac}^b \frac{\partial}{\partial m^b} = m^a m^c \Gamma_{ac}^1 \frac{\partial}{\partial m^1} + m^a m^c \Gamma_{ac}^2 \frac{\partial}{\partial m^2} + m^a m^c \Gamma_{ac}^3 \frac{\partial}{\partial m^3}$$

$$= (m^0 m^0 \Gamma_{00}^1 + m^2 m^2 \Gamma_{22}^1 + m^3 m^3 \Gamma_{33}^1 + m^1 m^0 \Gamma_{10}^1) \frac{\partial}{\partial m^1}$$

$$+ (m^3 m^3 \Gamma_{33}^2 + m^2 m^0 \Gamma_{20}^2 + m^2 m^1 \Gamma_{21}^2) \frac{\partial}{\partial m^2} + (m^3 m^0 \Gamma_{30}^3 + m^3 m^1 \Gamma_{31}^3 + m^3 m^2 \Gamma_{32}^3) \frac{\partial}{\partial m^3}$$

These coefficients become:

$$m^0 m^0 \Gamma_{00}^1 + m^2 m^2 \Gamma_{22}^1 + m^3 m^3 \Gamma_{33}^1 + m^1 m^0 \Gamma_{10}^1 = v^2 [D_r \Phi + \mu D_t \Lambda - (1-\mu^2) \Gamma/R]$$

$$m^3 m^3 \Gamma_{33}^2 + m^2 m^1 \Gamma_{21}^2 + m^2 m^0 \Gamma_{20}^2 = (1-\mu^2)^{1/2} \frac{v^2}{R} [-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi]$$

$$m^3 m^0 \Gamma_{30}^3 + m^3 m^1 \Gamma_{31}^3 + m^3 m^2 \Gamma_{32}^3 = (1-\mu^2)^{1/2} \frac{v^2}{R} \sin \Phi [U + \mu \Gamma + \mu \cos \Phi]$$

Then using

$$\frac{\partial}{\partial m^1} = \mu \frac{\partial}{\partial v} + \frac{(1-\mu^2)}{v} \frac{\partial}{\partial \mu}$$

$$\frac{\partial}{\partial m^2} = (1-\mu^2)^{1/2} \cos \Phi \left(\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right)$$

$$\frac{\partial}{\partial m^3} = (1-\mu^2)^{1/2} \sin \Phi \left(\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right)$$

we have

$$v^{-1} m^a m^c \Gamma_{ac}^b \frac{\partial}{\partial m^b} = v [D_r \Phi + \mu D_t \Lambda - (1-\mu^2) \frac{\Gamma}{R}] \left[\mu \frac{\partial}{\partial v} + \frac{(1-\mu^2)}{v} \frac{\partial}{\partial \mu} \right]$$

$$+ (1-\mu^2)^{1/2} \frac{v}{R} [-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi] (1-\mu^2)^{1/2} \cos \Phi \left(\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right)$$

$$+ (1-\mu^2)^{1/2} \frac{v}{R} \sin \Phi [U + \mu \Gamma + \mu \cos \Phi] (1-\mu^2)^{1/2} \sin \Phi \left(\frac{\partial}{\partial v} - \frac{\mu}{v} \frac{\partial}{\partial \mu} \right)$$

$$= v \left\{ \mu D_r \Phi + \mu^2 D_t \Lambda - \mu (1-\mu^2) \frac{\Gamma}{R} + (1-\mu^2) \cos \Phi [-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi] + \frac{(1-\mu^2)}{R} \sin^2 \Phi (U + \mu \Gamma + \mu \cos \Phi) \right\} \frac{\partial}{\partial v}$$

$$+ (1-\mu^2) \left\{ D_r \Phi + \mu D_t \Lambda - (1-\mu^2) \frac{\Gamma}{R} + \frac{\cos \Phi}{R} [-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi] (-\mu) + \frac{\sin^2 \Phi}{R} (U + \mu \Gamma + \mu \cos \Phi) (-\mu) \right\} \frac{\partial}{\partial \mu}$$

$$= v \left\{ \mu D_r \Phi + \mu^2 D_t \Lambda - \mu (1-\mu^2) \frac{\Gamma}{R} + \frac{(1-\mu^2)}{R} [\cos \Phi (-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi) + \sin^2 \Phi (U + \mu \Gamma + \mu \cos \Phi)] \right\} \frac{\partial}{\partial v}$$

$$+ (1-\mu^2) \left\{ D_r \Phi + \mu D_t \Lambda - (1-\mu^2) \frac{\Gamma}{R} - \frac{\mu}{R} [\cos \Phi (-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi) + \sin^2 \Phi (U + \mu \Gamma + \mu \cos \Phi)] \right\} \frac{\partial}{\partial \mu}$$

now $\cos \Phi (-\mu \sin^2 \Phi + (U + \mu \Gamma) \cos \Phi) + \sin^2 \Phi (U + \mu \Gamma + \mu \cos \Phi) = -\mu \sin^2 \Phi \cos \Phi + U \cos^2 \Phi + \mu \Gamma \cos^2 \Phi$

$$+ \mu \sin^2 \Phi \cos \Phi + \sin^2 \Phi U + \mu \Gamma \sin^2 \Phi = (U + \mu \Gamma)$$

$$\text{Hence } \gamma^i m^a m^c \Gamma_{ac}^b \frac{\partial}{\partial m^b} = \gamma \left\{ \mu D_r \Phi + \mu^2 D_\epsilon \Lambda - \mu (1-\mu^2) \frac{\Gamma}{R} + \frac{(1-\mu^2)}{R} (U + \mu \Gamma) \right\} \\ + (1-\mu^2) \left\{ D_r \Phi + \mu D_\epsilon \Lambda - (1-\mu^2) \frac{\Gamma}{R} - \frac{\mu}{R} (U + \mu \Gamma) \right\}$$

$$= \gamma \left\{ \mu D_r \Phi + \mu^2 D_\epsilon \Lambda + (1-\mu^2) \frac{U}{R} \right\} \frac{\partial}{\partial \gamma} + (1-\mu^2) \left\{ D_r \Phi + \mu D_\epsilon \Lambda - \frac{\Gamma}{R} - \mu \frac{U}{R} \right\} \frac{\partial}{\partial \mu}$$

and we then have

$$m^a \partial_a - m^a m^c \Gamma_{ac}^b \frac{\partial}{\partial m^b} = \gamma D_\epsilon + \mu \gamma D_r - \gamma^2 \left\{ \mu D_r \Phi + \mu^2 D_\epsilon \Lambda + (1-\mu^2) \frac{U}{R} \right\} \frac{\partial}{\partial \gamma} \\ + \gamma (1-\mu^2) \left\{ \frac{\Gamma}{R} - D_r \Phi + \mu \left(\frac{\Gamma}{R} - D_\epsilon \Lambda \right) \right\} \frac{\partial}{\partial \mu}$$

S(6) \Rightarrow

$$\frac{1}{\gamma} (e-a\partial) = D_\epsilon \downarrow + \mu D_r \downarrow - \gamma \left\{ \mu D_r \Phi + \mu^2 D_\epsilon \Lambda + (1-\mu^2) \frac{U}{R} \right\} \frac{\partial}{\partial \gamma} + (1-\mu^2) \left\{ \frac{\Gamma}{R} - D_r \Phi + \mu \left(\frac{\Gamma}{R} - D_\epsilon \Lambda \right) \right\} \frac{\partial}{\partial \mu}$$

Take (t', r, θ, ϕ) = inertial frame coordinates (t, M_r, θ, ϕ) = Lagrangian coordinates

$$M_r(r, t') = \int_0^r 4\pi(r')^2 \rho_0(r', t') dr'$$

$$t(r, t') = t' - c^2 \int_0^r v(r', t') dr'$$

and $v \equiv \frac{dr}{dt'} = - (4\pi r^2 \rho_0)^{-1} \left. \frac{\partial M_r}{\partial t'} \right|_r \Rightarrow \frac{\partial M_r}{\partial t'} = - (4\pi r^2 \rho_0) v$

$$dM_r = \frac{\partial M_r}{\partial r} dr + \frac{\partial M_r}{\partial t'} dt' = 4\pi r^2 \rho_0(r, t) dr + (-4\pi r^2 \rho_0 v) dt'$$

(95.64) $\frac{dM_r}{4\pi r^2 \rho_0} = dr - v dt' \equiv dx$

$$dt = \frac{\partial t}{\partial t'} dt' + \frac{\partial t}{\partial r} dr = dt' - \frac{1}{c^2} v(r, t) dr - \frac{1}{c^2} \left[\int_0^r \frac{\partial v(r', t')}{\partial t'} dr' \right] dt'$$

(95.66) now define $I \equiv \int_0^r (\partial v / \partial t') dr'$, then

(95.65) $dt = dt' (1 - I/c^2) - (v/c^2) dr$

Solve for dr and dt' :

$$dr = dx + v dt'$$

$$\Rightarrow dt = dt' (1 - I/c^2) - (v/c^2) (dx + v dt')$$

$$= dt' (1 - I/c^2) - (v^2/c^2) dt' - (v/c^2) dx$$

$$dt' = [dt + (v/c^2) dx] / [1 - I/c^2 - v^2/c^2]$$

(95.68) $dt' = D^{-1} [dt + (v/c^2) dx]$

$$dr = dx + v D^{-1} [dt + (v/c^2) dx]$$

$$= dx (1 + D^{-1} v^2/c^2) + D^{-1} v dt$$

$$= dx \left[\frac{1 - I/c^2 - v^2/c^2 + v^2/c^2}{1 - I/c^2 - v^2/c^2} \right] D^{-1} + D^{-1} v dt$$

(95.67) $dr = D^{-1} (1 - I/c^2) dx + D^{-1} v dt$

(95.68) $D = 1 - I/c^2 - v^2/c^2$

Substitute into the inertial frame metric

$$ds^2 = -c^2 (dt')^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$= -c^2 [dt + v dx/c^2]^2 / D^2 + [(1 - I/c^2) dx + v dt]^2 / D^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$ds^2 = -c^2 D^{-2} dt^2 - (v^2/c^2) D^{-2} dx^2 - 2v dx dt D^{-2} + v^2 D^2 dt^2 + (1 - I/c^2) D^2 dx^2 + 2v(1 - I/c^2) dx dt D^{-2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(95.71) almost $ds^2 = (v^2/c^2) D^{-2} dt^2 + [(1 - I/c^2)^2 - v^2/c^2] D^2 dx^2 - (2vI/c^2) D^{-2} dx dt + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
 $= -G dt^2 + F dx^2 - 2H dx dt + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

now $G = (v^2/c^2) D^{-2} = -c^2(1 - v^2/c^2) / [1 - (I+v^2)/c^2]^2 \cong -c^2 / [1 - 2(I+v^2)/c^2 + (I+v^2)^2/c^4]$
 $\rightarrow O(v/c)$
 $-G \cong -c^2 / [1 - 2I/c^2] \cong -c^2(1 + 2I/c^2) \cong -c^2 - 2I$
 (95.73) (95.75)

(95.72) $F = [(1 - I/c^2)^2 - v^2/c^2] / [1 - (I+v^2)/c^2]^2 = \frac{[1 - 2I/c^2 - v^2/c^2 + I^2/c^4]}{1 - 2I/c^2 - 2v^2/c^2 + \frac{I^2}{c^4} + \frac{2Iv^2}{c^4} + \frac{v^4}{c^4}}$

$$F \cong \frac{(1 - 2I/c^2)}{(1 - 2I/c^2)} + O(v^2/c^2) = 1 + O(v^2/c^2)$$

(95.74) $2vI/c^2 = 2v \int \frac{\partial v}{\partial t} dr' / c^2 = O(v^2/c^2) = 2H(4\pi r^2 \rho_0)$

and $dx = dm_r / 4\pi r^2 \rho_0$

Thus we have the $O(v/c)$ metric

(95.71) $ds^2 = (-c^2 - 2I) dt^2 + \frac{dm_r^2}{(4\pi \rho_0 r^2)^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$

Can compare this with Castor's metric eq (18) of Castor 1972.

Compare this with:

(95.45) $ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$ $dr \rightarrow dM_r$

to get

(95.76)
$$\left. \begin{aligned} e^{2\Phi} &= c^2 + 2I \Rightarrow \Phi = \frac{1}{2} \ln(c^2 + 2I) \\ e^{2\Lambda} &= (4\pi r^2 \rho_0)^{-2} \Rightarrow 2\Lambda = -2 \ln(4\pi r^2 \rho_0) \\ \Lambda &= -\ln(4\pi r^2 \rho_0) \\ R &= r \\ dr &= dM_r \end{aligned} \right\}$$

Then calculate derivatives

$$(95.77) \quad D_t \equiv e^{-\frac{\Phi}{c}} \frac{\partial}{\partial t} = (c^2 + 2I)^{1/2} \frac{\partial}{\partial t} = \frac{1}{c(1+2I/c^2)^{1/2}} \frac{\partial}{\partial t} \\ = \frac{1}{c} \left(1 - \frac{I}{c^2}\right) \frac{\partial}{\partial t} \approx \frac{1}{c} \frac{\partial}{\partial t}$$

$$(95.77) \quad D_r = e^{-\frac{\Phi}{c}} \frac{\partial}{\partial r} = (4\pi r^2 \rho_0) \frac{\partial}{\partial m_r} = \frac{\partial}{\partial r}$$

Then

$$U = D_t R = \frac{1}{c} \frac{\partial R}{\partial t} = v/c$$

$$\Gamma = D_r R = \frac{\partial R}{\partial r}(r) = 1$$

$$(95.78) \quad D_r \Phi = D_r \left[\frac{1}{2} \ln(c^2 + 2I) \right] = \frac{1}{2} (c^2 + 2I)^{-1} \frac{\partial}{\partial r} (2I) = (c^2 + 2I)^{-1} \frac{\partial}{\partial r} \int_0^r \frac{\partial v}{\partial t'} dr' \\ = (c^2 + 2I)^{-1} \frac{\partial v}{\partial t'} = \frac{1}{c^2} \frac{\partial v}{\partial t'} - \frac{1}{c^2} (2I/c^2) \frac{\partial v}{\partial t'} \approx \frac{1}{c^2} \frac{\partial v}{\partial t'} + O(v^2/c^2) \\ = a/c^2$$

$$(95.79) \quad D_t \Lambda = \frac{1}{c} \frac{\partial}{\partial t} [-\ln(4\pi r^2 \rho_0)] = -\frac{1}{c} \frac{1}{(4\pi r^2 \rho_0)} \frac{\partial}{\partial t} (4\pi r^2 \rho_0) = -\frac{1}{c r^2 \rho_0} \frac{\partial}{\partial t} (r^2 \rho_0) \\ \frac{\partial}{\partial t} \leftrightarrow \frac{D}{D t} ?? \\ = -\frac{1}{c} \frac{1}{r^2 \rho_0} \left(2r \frac{\partial r}{\partial t} \rho_0 + r^2 \frac{\partial \rho_0}{\partial t} \right) = -\frac{1}{c} \left[\frac{2v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right]$$

Now return to (95.61):

$$r'(e-a\mathcal{J}) = D_t \mathcal{J} + \mu D_r \mathcal{J} - v \left[\mu D_r \Phi + \mu^2 D_t \Lambda + (1-\mu^2) \frac{U}{R} \right] \frac{\partial \mathcal{J}}{\partial v} + (1-\mu^2) \left\{ \frac{\Gamma}{R} - D_r \Phi + \mu \left(\frac{v}{R} - D_t \Lambda \right) \right\} \frac{\partial \mathcal{J}}{\partial \mu}$$

and first substitute in (95.78)-(95.79)

$$r'(e-a\mathcal{J}) = c^{-1} \frac{\partial \mathcal{J}}{\partial t} \left(\frac{D\mathcal{J}}{Dt} \right) + \mu \frac{\partial \mathcal{J}}{\partial r} - v \left[\mu \frac{a}{c^2} - \frac{\mu^2}{c} \left(\frac{2v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + (1-\mu^2) \frac{v}{rc} \right] \frac{\partial \mathcal{J}}{\partial v} \\ + (1-\mu^2) \left\{ \frac{1}{r} - \frac{a}{c^2} + \mu \left(\frac{v}{cr} + \frac{2v}{rc} + \frac{1}{c} \frac{\partial \ln \rho_0}{\partial t} \right) \right\} \frac{\partial \mathcal{J}}{\partial \mu}$$

compare w/ astor (1972) eqn (20)

$$r'(e-a\mathcal{J}) = \frac{1}{c} \frac{\partial \mathcal{J}}{\partial t} + \mu \frac{\partial \mathcal{J}}{\partial r} - v \left[\mu \frac{a}{c^2} - \frac{\mu^2}{c} \frac{\partial \ln \rho_0}{\partial t} + (1-3\mu^2) \frac{v}{rc} \right] \frac{\partial \mathcal{J}}{\partial v} \\ + (1-\mu^2) \left\{ \frac{\mu}{c} \left(\frac{3v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial \mathcal{J}}{\partial \mu}$$

Now replace: $\mathcal{J} = I_0/v_0^3$ $e = \eta_0/v_0^2$ $a = v_0 \chi_0$ and explicitly note that μ is μ_0 , v is v_0

$$r_0^{-1} (\eta_0/v_0^2 - v_0 \chi_0 I_0/v_0^3) = v_0^3 (\eta_0 - \chi_0 I_0) = \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{I_0}{v_0^3} \right) + \mu_0 \frac{\partial}{\partial r} \left(\frac{I_0}{v_0^3} \right) - v_0 \left[\mu_0 \frac{a}{c^2} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + (1-3\mu_0^2) \frac{v_0}{rc} \right] \frac{\partial}{\partial v_0} \left(\frac{I_0}{v_0^3} \right) \\ + (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v_0}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial}{\partial \mu_0} \left(\frac{I_0}{v_0^3} \right)$$

$$\gamma_0^{-3}(\eta_0 - \chi_0 I_0) = \gamma_0^{-3} \left(\frac{1}{c} \frac{\partial I_0}{\partial t} + \mu_0 \frac{\partial I_0}{\partial r} \right) - \gamma_0^{-2} \left[\mu_0 \frac{a}{c^2} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{(1-3\mu_0^2)v}{rc} \right] \frac{\partial I_0}{\partial v_0} + \frac{3v_0}{v_0^4} \left[\frac{\mu_0 a}{c^2} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{(1-3\mu_0^2)v}{rc} \right] I_0$$

$$+ \gamma_0^{-3} (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0}$$

and thus

$$\eta_0 - \chi_0 I_0 = \frac{1}{c} \frac{\partial I_0}{\partial t} + \mu_0 \frac{\partial I_0}{\partial r} + (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} - \gamma_0 \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{\mu_0 a}{c^2} \right\} \frac{\partial I_0}{\partial v_0}$$

$$+ 3 \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{\mu_0 a}{c^2} \right\} I_0$$

note also D_r can be written $4\pi r^2 \rho_0 \frac{\partial}{\partial m_r}$, yielding $\mu D_r \mu = 4\pi r^2 \rho_0 \mu_0 \frac{\partial}{\partial m_r} = \frac{4\pi r^2 \rho_0 \mu_0}{v_0^3} \frac{\partial I_0}{\partial m_r}$
hence

(9.5.80)
$$\eta_0 - \chi_0 I_0 = \frac{1}{c} \frac{\partial I_0}{\partial t} + 4\pi r^2 \rho_0 \mu_0 \frac{\partial I_0}{\partial m_r} + (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{\partial \ln \rho_0}{\partial t} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} - \gamma_0 \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{\mu_0 a}{c^2} \right\} \frac{\partial I_0}{\partial v_0}$$

compare Castor (21)
$$+ 3 \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{\partial \ln \rho_0}{\partial t} + \frac{\mu_0 a}{c^2} \right\} I_0$$

Where the derivatives $\frac{\partial}{\partial t}$ are Lagrangian derivatives, hence can be written $\frac{D}{Dt}$ as in DMM eqn (9.5.80)

The continuity eqn gives $\frac{D \ln \rho_0}{Dt} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v) = -\frac{2v}{r} - \frac{\partial v}{\partial r}$ and thus

re retrieve (9.5.12)
$$\eta_0 - \chi_0 I_0 = \frac{1}{c} \frac{D I_0}{Dt} + \mu_0 \frac{\partial I_0}{\partial r} + (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{v}{r} - \frac{\partial v}{\partial r} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} - \gamma_0 \left\{ \frac{(1-\mu_0^2)v}{rc} + \frac{\mu_0^2}{c} \frac{\partial v}{\partial r} + \frac{\mu_0 a}{c^2} \right\} \frac{\partial I_0}{\partial v_0}$$

$$+ 3 \left\{ \frac{(1-\mu_0^2)v}{rc} + \frac{\mu_0^2}{c} \frac{\partial v}{\partial r} + \frac{\mu_0 a}{c^2} \right\} I_0$$

Integrate this equation over $d\mu_0$ and over $\mu_0 d\mu_0$ and with $(J_0, H_0, K_0) \rightarrow (E_0, F_0, P_0)$
 $N_0 \rightarrow Q_0$

$$\int (\eta_0 - \chi_0 I_0) d\mu_0 = \frac{1}{c} \int \frac{D I_0}{Dt} d\mu_0 + 4\pi r^2 \rho_0 \int \mu_0 \frac{\partial I_0}{\partial m_r} d\mu_0 + \int (1-\mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{D \ln \rho_0}{Dt} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} d\mu_0$$

$$- \gamma_0 \int \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{D \ln \rho_0}{Dt} + \frac{\mu_0 a}{c^2} \right\} \frac{\partial I_0}{\partial v_0} d\mu_0 + 3 \int \left\{ \frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c} \frac{D \ln \rho_0}{Dt} + \frac{\mu_0 a}{c^2} \right\} I_0 d\mu_0$$

$$\eta_0 - \chi_0 J_0(v_0) = \frac{1}{c} \frac{D J_0(v_0)}{Dt} + 4\pi r^2 \rho_0 \frac{\partial H_0}{\partial m_r} - \int \frac{\partial}{\partial \mu_0} \left\{ (1-\mu_0^2) \left[\frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{D \ln \rho_0}{Dt} \right) + \frac{1}{r} - \frac{a}{c^2} \right] \right\} I_0 d\mu_0$$

$$- \gamma_0 \int \left\{ \frac{v}{rc} \frac{\partial I_0}{\partial v_0} - \frac{3v}{rc} \frac{\mu_0^2}{c} \frac{\partial I_0}{\partial v_0} - \frac{D \ln \rho_0}{Dt} \frac{\mu_0^2}{c} \frac{\partial I_0}{\partial v_0} + \frac{a}{c^2} \mu_0 \frac{\partial I_0}{\partial v_0} \right\} d\mu_0 + \int \left\{ \frac{3v}{rc} I_0 - \frac{9\mu_0^2 v}{rc} I_0 - \frac{3\mu_0^2}{c} \frac{D \ln \rho_0}{Dt} I_0 + \frac{3\mu_0 a}{c^2} I_0 \right\} d\mu_0$$

$$\begin{aligned} \gamma_0 - \chi_0 J_0 &= c^{-1} \frac{\partial J_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 H_0) - 4\pi p_0 H_0 \cdot 2r \left\{ (1 - \mu_0^2) \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) J_0 + (-2\mu_0) \left[\frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) + \frac{1}{r} - \frac{a}{c^2} \right] J_0 \right\} \\ &\quad - \gamma_0 \left\{ \frac{v}{cr} \frac{\partial J_0}{\partial v_0} - \frac{3v}{cr} \frac{\partial K_0}{\partial v_0} - \frac{1}{c} \frac{D\mu_0}{Dt} \frac{\partial K_0}{\partial v_0} + \frac{a}{c^2} \frac{\partial H_0}{\partial v_0} \right\} + \left\{ \frac{3v}{cr} J_0 - \frac{9v}{cr} K_0 - \frac{3}{c} \frac{D\mu_0}{Dt} K_0 + \frac{3a}{c^2} H_0 \right\} \\ &= c^{-1} \frac{\partial J_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 H_0) - 4\pi \cdot 2rp_0 H_0 - \left\{ \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) J_0 - \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) K_0 - \frac{2}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) K_0 - \frac{2}{r} H_0 + \frac{2a}{c^2} H_0 \right\} \\ &\quad - \gamma_0 \left\{ \frac{v}{cr} \frac{\partial J_0}{\partial v_0} - \frac{3v}{cr} \frac{\partial K_0}{\partial v_0} - \frac{1}{c} \frac{D\mu_0}{Dt} \frac{\partial K_0}{\partial v_0} + \frac{a}{c^2} \frac{\partial H_0}{\partial v_0} \right\} + \left\{ \frac{3v}{cr} J_0 - \frac{9v}{cr} K_0 - \frac{3}{c} \frac{D\mu_0}{Dt} K_0 + \frac{3a}{c^2} H_0 \right\} \end{aligned}$$

Non-derivative terms are:

$$\begin{aligned} &-4\pi(2rp_0 H_0) - \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) J_0 + \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) K_0 + \frac{6v}{cr} K_0 + \frac{2}{c} \frac{D\mu_0}{Dt} K_0 + \frac{2H_0}{r} + \frac{2a}{c^2} H_0 \\ &\quad + \frac{3v}{cr} J_0 - \frac{9v}{cr} K_0 - \frac{3}{c} \frac{D\mu_0}{Dt} K_0 + \frac{3a}{c^2} H_0 \\ &= -4\pi(2rp_0 H_0) - \frac{1}{c} \frac{D\mu_0}{Dt} J_0 + \frac{3v}{cr} K_0 + \frac{6v}{cr} K_0 - \frac{9v}{cr} K_0 + \frac{1}{c} \frac{D\mu_0}{Dt} K_0 + \frac{2}{c} \frac{D\mu_0}{Dt} K_0 - \frac{3}{c} \frac{D\mu_0}{Dt} K_0 \\ &\quad + \frac{2H_0}{r} - \frac{2a}{c^2} H_0 + \frac{3a}{c^2} H_0 \\ &= -\frac{1}{c} \frac{D\mu_0}{Dt} J_0 - 4\pi(2rp_0 H_0) + \frac{2H_0}{r} + \frac{a}{c^2} H_0 \end{aligned}$$

$$\begin{aligned} \gamma_0 - \chi_0 J_0 &= \frac{1}{c} \frac{\partial J_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 H_0) - \gamma_0 \left\{ \frac{v}{cr} \frac{\partial J_0}{\partial v_0} - \left(\frac{3v}{cr} + \frac{1}{c} \frac{D\mu_0}{Dt} \right) \frac{\partial K_0}{\partial v_0} + \frac{a}{c^2} \frac{\partial H_0}{\partial v_0} \right\} - \frac{1}{c} \frac{D\mu_0}{Dt} J_0 - 4\pi(2rp_0 H_0) \\ &\quad + \frac{2H_0}{r} + \frac{a}{c^2} H_0 \\ &= \frac{1}{c} \frac{\partial J_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 H_0) - \frac{\partial}{\partial v_0} \left\{ \gamma_0 \left(\frac{v}{cr} \right) (J_0 - 3K_0) - \frac{1}{c} \frac{D\mu_0}{Dt} K_0 + \frac{a}{c^2} H_0 \right\} \\ &\quad - \frac{1}{c} \frac{D\mu_0}{Dt} J_0 - \frac{4\pi(2rp_0 H_0)}{4\pi r p_0} + \frac{2H_0}{r} + \frac{a}{c^2} H_0 \\ &\quad + \left\{ \frac{v}{cr} \left(\frac{\partial J_0}{\partial v_0} - 3K_0 \right) - \frac{1}{c} \frac{D\mu_0}{Dt} K_0 + \frac{a}{c^2} H_0 \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{c} \frac{\partial J_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 H_0) - \frac{\partial}{\partial v_0} \left\{ \gamma_0 \left[\frac{v}{cr} (J_0 - 3K_0) - \frac{1}{c} \frac{D\mu_0}{Dt} K_0 + \frac{a}{c^2} H_0 \right] \right\} \\ &\quad + \frac{v}{cr} (J_0 - 3K_0) - \frac{1}{c} \frac{D\mu_0}{Dt} (J_0 + K_0) - \frac{4\pi(2rp_0 H_0)}{4\pi r p_0} + \frac{2H_0}{r} + \frac{2a}{c^2} H_0 \end{aligned}$$

Replace $E_0 = \frac{4\pi}{c} J_0$, $F_0 = 4\pi H_0$ and $P_0 = \frac{4\pi}{c} K_0 \Rightarrow J_0 = \frac{c}{4\pi} E_0$, $H_0 = \frac{1}{4\pi} F_0$, $K_0 = \frac{c}{4\pi} P_0$

$$\begin{aligned} \gamma_0 - \chi_0 J_0 &= \frac{1}{4\pi} \frac{\partial E_0}{\partial t} + p_0 \frac{\partial}{\partial r} (r^2 F_0) - \frac{\partial}{\partial v_0} \left\{ \frac{c\gamma_0}{4\pi} \left[\frac{v}{cr} (E_0 - 3P_0) - \frac{1}{c} \frac{D\mu_0}{Dt} P_0 \right] + \frac{a}{c^2} \frac{v_0}{4\pi} F_0 \right\} \\ &\quad + \frac{v}{cr} \left(\frac{c}{4\pi} (E_0 - 3P_0) \right) - \frac{1}{c} \frac{D\mu_0}{Dt} \left(\frac{c}{4\pi} (E_0 + P_0) \right) + \frac{2a}{c^2} \frac{F_0}{4\pi} \end{aligned}$$

$$\begin{aligned} 4\pi \gamma_0 - \chi_0 E_0 c &= \frac{\partial E_0}{\partial t} + 4\pi p_0 \frac{\partial}{\partial r} (r^2 F_0) - \frac{\partial}{\partial v_0} \left\{ \gamma_0 \left[\frac{v}{r} (E_0 - 3P_0) - \frac{D\mu_0}{Dt} P_0 + \frac{a}{c^2} F_0 \right] \right\} \\ &\quad + \frac{v}{r} (E_0 - 3P_0) - \frac{D\mu_0}{Dt} (E_0 + P_0) + \frac{2a}{c^2} F_0 \end{aligned}$$

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3. final equation is:

(95.82)

$$4\pi\epsilon_0 - c\chi_0 E_0 = \frac{DE_0}{Dt} + 4\pi\epsilon_0 \frac{\partial}{\partial r} (r^2 F_0) - \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{r} (E_0 - 3P_0) - \frac{D\mu_0}{Dt} P_0 + \frac{a}{c^2} F_0 \right] \right\} + \frac{v}{r} (E_0 - 3P_0) - \frac{D\mu_0}{Dt} (E_0 + P_0) + \frac{2a}{c^2} F_0$$

Now integrate (95.80) over $\mu_0 d\mu_0$

$$\int (\epsilon_0 - \chi_0 I_0) \mu_0 d\mu_0 = \frac{1}{c} \int \mu_0 d\mu_0 \frac{DE_0}{Dt} + 4\pi\epsilon_0 \int \mu_0^2 d\mu_0 \frac{\partial I_0}{\partial r} + \int \mu_0 d\mu_0 (1 - \mu_0^2) \left\{ \frac{\mu_0}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) + \frac{1}{r} - \frac{a}{c^2} \right\} \frac{\partial F_0}{\partial \mu_0} - \chi_0 \int \left\{ \frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c} \frac{D\mu_0}{Dt} + \mu_0 \frac{a}{c^2} \right\} \frac{\partial I_0}{\partial \mu_0} \mu_0 d\mu_0 + 3 \int \mu_0 d\mu_0 \left\{ \frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c} \frac{D\mu_0}{Dt} + \frac{\mu_0 a}{c^2} \right\} I_0$$

$$\Rightarrow -\chi_0 H_0 = \frac{1}{c} \frac{DH_0}{Dt} + 4\pi\epsilon_0 \frac{\partial K_0}{\partial r} + \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) \left[\int (\mu_0^2 - \mu_0^4) d\mu_0 \frac{\partial I_0}{\partial \mu_0} \right] + \left(\frac{1}{r} - \frac{a}{c^2} \right) \int (\mu_0 - \mu_0^3) \frac{\partial I_0}{\partial \mu_0} d\mu_0$$

$$-v_0 \frac{\partial}{\partial y_0} \left\{ \frac{v}{cr} \left(\frac{\partial H_0}{\partial y_0} - 3 \frac{\partial N_0}{\partial y_0} \right) + \frac{3v}{c} \frac{D\mu_0}{Dt} \frac{\partial N_0}{\partial y_0} + \frac{a}{c^2} \frac{\partial K_0}{\partial y_0} \right\} + \frac{3v}{cr} (H_0 - 3N_0) - \frac{3D\mu_0}{c} N_0 + \frac{3a}{c^2} K_0$$

$$-\chi_0 H_0 = \frac{1}{c} \frac{DH_0}{Dt} + 4\pi\epsilon_0 \frac{\partial K_0}{\partial r} + \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) \left(- \int I_0 (2\mu_0 - 4\mu_0^3) d\mu_0 + \left(\frac{1}{r} - \frac{a}{c^2} \right) \int I_0 (1 - 3\mu_0^2) d\mu_0 \right)$$

$$-v_0 \frac{\partial}{\partial y_0} \left\{ \frac{v}{cr} (H_0 - 3N_0) - \frac{1}{c} \frac{D\mu_0}{Dt} N_0 + \frac{a}{c^2} K_0 \right\} + \frac{3v}{cr} (H_0 - 3N_0) - \frac{3}{c} \frac{D\mu_0}{Dt} N_0 + \frac{3a}{c^2} K_0$$

and thus $-\chi_0 H_0 = \frac{1}{c} \frac{DH_0}{Dt} + 4\pi\epsilon_0 r^2 \frac{\partial K_0}{\partial r} + \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) (4N_0 - 2H_0) + \left(\frac{1}{r} - \frac{a}{c^2} \right) (3K_0 - \dot{J}_0)$

$$-v_0 \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{cr} (H_0 - 3N_0) - \frac{1}{c} \frac{D\mu_0}{Dt} N_0 + \frac{a}{c^2} K_0 \right] \right\} + \frac{v}{cr} (H_0 - 3N_0) - \frac{1}{c} \frac{D\mu_0}{Dt} N_0 + \frac{a}{c^2} K_0 + \frac{3v}{cr} (H_0 - 3N_0) - \frac{3}{c} \frac{D\mu_0}{Dt} N_0 + \frac{3a}{c^2} K_0$$

$$= \frac{1}{c} \frac{DH_0}{Dt} + 4\pi\epsilon_0 \frac{\partial K_0}{\partial r} - \frac{4\pi\epsilon_0 K_0}{4\pi r^2} + \left(\frac{a}{c^2} - \frac{1}{r} \right) \dot{J}_0 + \left(\frac{1}{c} \right) \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) (2H_0) + \left(\frac{1}{r} - \frac{a}{c^2} \right) 3K_0 + \frac{1}{c} \left(\frac{3v}{r} + \frac{D\mu_0}{Dt} \right) \cdot 4N_0 - \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{cr} (H_0 - 3N_0) - \frac{1}{c} \frac{D\mu_0}{Dt} N_0 + \frac{a}{c^2} K_0 \right] \right\} + \frac{4v}{cr} H_0 + \frac{3v}{c^2} K_0 + \left(-\frac{3v}{cr} N_0 - \frac{3v}{cr} N_0 - \frac{4}{c} \frac{D\mu_0}{Dt} N_0 + \frac{3a}{c^2} K_0 \right)$$

$$-\chi_0 H_0 = \frac{1}{c} \frac{DH_0}{Dt} + 4\pi\epsilon_0 \frac{\partial (r^2 K_0)}{\partial r} + \left(\frac{a}{c^2} - \frac{1}{r} \right) \dot{J}_0 + \left(\frac{2v}{cr} + \frac{3}{c} \frac{D\mu_0}{Dt} \right) H_0 + \frac{K_0}{r} + \frac{a}{c^2} K_0 - \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{cr} (H_0 - 3N_0) - \frac{1}{c} \frac{D\mu_0}{Dt} N_0 + \frac{a}{c^2} K_0 \right] \right\}$$

Substitute $E_0 = \frac{4\pi}{c} J_0$ $F_0 = 4\pi H_0$ $P_0 = \frac{4\pi}{c} K_0$ $Q_0 = \frac{4\pi}{c} N_0$

$$-\chi_0 \frac{F_0}{4\pi} = \frac{1}{4\pi c} \frac{DF_0}{Dt} + \frac{4\pi\epsilon_0 c}{4\pi} \frac{\partial}{\partial r} (r^2 P_0) - \left(\frac{2v}{cr} + \frac{3}{c} \frac{D\mu_0}{Dt} \right) \frac{F_0}{4\pi} + \frac{c}{4\pi} P_0 - \frac{a}{c^2} \frac{F_0}{4\pi} + \frac{c}{4\pi} \left(\frac{a}{c^2} - \frac{1}{r} \right) E_0 - \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{cr} (F_0 - 3Q_0) - \frac{1}{c} \frac{D\mu_0}{Dt} Q_0 + \frac{a}{c^2} P_0 \right] \right\}$$

or

$$-\chi_0 F_0 = \frac{1}{c} \frac{DF_0}{Dt} + 4\pi\epsilon_0 c \frac{\partial}{\partial r} (r^2 P_0) - \left(\frac{2v}{cr} + \frac{3}{c} \frac{D\mu_0}{Dt} \right) F_0 + \frac{c}{r} P_0 + \frac{a}{c^2} P_0 - \frac{\partial}{\partial y_0} \left\{ v_0 \left[\frac{v}{cr} (F_0 - 3Q_0) - \frac{1}{c} \frac{D\mu_0}{Dt} Q_0 + \frac{a}{c^2} P_0 \right] \right\} + c \left(\frac{a}{c^2} - \frac{1}{r} \right) E_0$$

$$-\chi_0 F_0 = \frac{1}{c} \frac{DF_0}{Dt} + 4\pi p_0 c \frac{\partial}{\partial M_r} (r^2 P_0) - \frac{2}{c} \left(\frac{v}{r} + \frac{D \ln p_0}{Dt} \right) F_0 + \frac{c P_0}{r} + \frac{a}{c^2} (P_0 + E_0) - \frac{c E_0}{r} - \frac{2}{\partial v_0} \left\{ v_0 \left[\frac{v}{c r} (F_0 - 3Q_0) - \frac{D \ln p_0}{Dt} Q_0 + \frac{a}{c^2} P_0 \right] \right\}$$

~~Multiply~~ Divide by c

$$-c \chi_0 F_0 = \frac{1}{c^2} \frac{DF_0}{Dt} + 4\pi p_0 \frac{\partial}{\partial M_r} (r^2 P_0) + \frac{1}{r} (P_0 - E_0) + \frac{a}{c^2} (P_0 + E_0) - \frac{2}{c^2} \left(\frac{v}{r} + \frac{D \ln p_0}{Dt} \right) F_0 - \frac{2}{c^2 \partial v_0} \left\{ v_0 \left[\frac{v}{c r} (F_0 - 3Q_0) - \frac{D \ln p_0}{Dt} Q_0 + \frac{a}{c^2} P_0 \right] \right\}$$

$$(5.83) \quad -\frac{\chi_0 F_0}{c} = \frac{1}{c^2} \frac{DF_0}{Dt} + 4\pi p_0 \frac{\partial}{\partial M_r} (r^2 P_0) + \frac{1}{r} (P_0 - E_0) + \frac{a}{c^2} (P_0 + E_0) - \frac{2}{c^2} \left(\frac{v}{r} + \frac{D \ln p_0}{Dt} \right) F_0 + \frac{2}{c^2 \partial v_0} \left\{ v_0 \left[\frac{v}{c r} (3Q_0 - F_0) + \frac{D \ln p_0}{Dt} Q_0 - a P_0 \right] \right\}$$

Integrate (45.82) over frequency:

$$\int (4\pi p_0 - c \chi_0 E_0) dv_0 = \int dv_0 \frac{DF_0}{Dt} + 4\pi p_0 \frac{\partial}{\partial M_r} (r^2 \int F_0 dv_0) - \int \frac{2}{\partial v_0} \left\{ \right\} dv_0 + \frac{v}{r} \int (E_0 - 3P_0) dv_0 - \frac{D \ln p_0}{Dt} \int (E_0 + P_0) dv_0 + \frac{2a}{c^2} \int F_0 dv_0$$

$$(45.84) \quad \int (4\pi p_0 - c \chi_0 E_0) dv_0 = \frac{DF_0}{Dt} + 4\pi p_0 \frac{\partial}{\partial M_r} (r^2 F_0) + \frac{v}{r} (E_0 - 3P_0) - \frac{D \ln p_0}{Dt} (E_0 + P_0) + \frac{2a}{c^2} F_0$$

Integrate (45.83) over frequency:

$$(45.85) \quad -\frac{1}{c} \int \chi_0 F_0 dv_0 = \frac{1}{c^2} \frac{DF_0}{Dt} + 4\pi p_0 \frac{\partial}{\partial M_r} (r^2 P_0) + \frac{1}{r} (P_0 - E_0) + \frac{a}{c^2} (P_0 + E_0) - \frac{2}{c^2} \left(\frac{v}{r} + \frac{D \ln p_0}{Dt} \right) F_0$$

Relativistic momentum eqn in comoving frame

$$\rho_{000} \frac{Dv}{Dt} = \underline{f} - \underline{\nabla} P + \underline{G}_0$$

take nonrelativistic limit in spherically symmetric case:

$$\rho_{000} \rightarrow \rho_0$$

$$\rho_0 \frac{Dv}{Dt} = - \frac{GM_{r\rho_0}}{r^2} - \frac{\partial P}{\partial r} + \frac{1}{c} \int \chi_0(v_0) F_0(v_0) dv_0$$

now from (95.85),

$$- \frac{1}{c} \int \chi_0(v_0) F_0(v_0) dv_0 = \frac{1}{c^2} \frac{DF_0}{Dt} + 4\pi \rho_0 r^2 \frac{\partial P_0}{\partial M_r} + \frac{3P_0 - E_0}{r} - \frac{z}{c^2} \left(\frac{v}{r} + \frac{D \ln \rho_0}{Dt} \right) F_0 + \frac{a}{c^2} (E_0 + P_0)$$

Drop the $\frac{a}{c^2}$ terms and write $\partial M_r = 4\pi \rho_0 r^2 dr$ to obtain

$$- \frac{1}{c} \int \chi_0(v_0) F_0(v_0) dv_0 = \frac{1}{c^2} \frac{DF_0}{Dt} + \frac{\partial P_0}{\partial r} + \frac{3P_0 - E_0}{r} - \frac{z}{c^2} \left(\frac{v}{r} + \frac{D \ln \rho_0}{Dt} \right) F_0$$

then

$$\rho_0 \frac{Dv}{Dt} = - \frac{GM_{r\rho_0}}{r^2} - \frac{\partial P}{\partial r} - \frac{1}{c^2} \frac{DF_0}{Dt} - \frac{\partial P_0}{\partial r} - \frac{3P_0 - E_0}{r} + \left(\frac{2v}{c^2 r} + \frac{z}{c^2} \frac{D \ln \rho_0}{Dt} \right) F_0$$

$$\text{Now } \frac{D \ln \rho_0}{Dt} = - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v) = - \frac{2v}{r} - \frac{\partial v}{\partial r} \Rightarrow \frac{1}{c^2} \left(\frac{2v}{r} + z \frac{D \ln \rho_0}{Dt} \right) = \frac{1}{c^2} \left(\frac{2v}{r} - \frac{4v}{r} - \frac{\partial v}{\partial r} \right) = - \frac{1}{c^2} \left(\frac{2v}{r} - \frac{\partial v}{\partial r} \right)$$

thus

$$\rho_0 \frac{Dv}{Dt} + \frac{1}{c^2} \frac{DF_0}{Dt} - \frac{1}{c^2} \frac{D \ln \rho_0}{Dt} F_0 = - \frac{GM_{r\rho_0}}{r^2} - \frac{\partial}{\partial r} (P + P_0) - \frac{(3P_0 - E_0)}{r} + \left(\frac{2v}{c^2 r} - \frac{2v}{c^2 r} - \frac{1}{c^2} \frac{\partial v}{\partial r} \right) F_0$$

$$(96.4) \quad \rho_0 \frac{Dv}{Dt} + \frac{\rho_0}{c^2} \left[\frac{D}{Dt} \left(\frac{F_0}{\rho_0} \right) \right] = - \frac{GM_{r\rho_0}}{r^2} - \frac{\partial}{\partial r} (P + P_0) - \frac{(3P_0 - E_0)}{r} - \frac{1}{c^2} \frac{\partial v}{\partial r} F_0$$

On fluid-flow timescale $\frac{\rho_0}{c^2} \frac{D}{Dt} \left(\frac{F_0}{\rho_0} \right)$ and $\frac{1}{c^2} \frac{\partial v}{\partial r} F_0$ are $O(\frac{v}{c})$ wrt $\frac{\partial P_0}{\partial r}$ in both streaming limit and diffusion limit. We can then write:

$$\rho_0 \frac{Dv}{Dt} = \underline{f} - \underline{\nabla} P$$

Gas Energy Equation:

$$\rho_0 \left[\frac{D\epsilon}{Dt} + p \frac{D\rho_0^{-1}}{Dt} \right] = c (F_0^o + G_0^o)$$

$c F_0^o =$ rate of nonmechanical energy input $= \rho_0 \epsilon$

$$c G_0^o = \int_0^\infty \{ c \chi_0(v_0) E_0(v_0) - 4\pi \eta_0(v_0) \} dv_0$$

thus $\rho_0 \left[\frac{D\epsilon}{Dt} + p \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) \right] = \rho_0 \epsilon + \int [c \chi_0(v_0) E_0(v_0) - 4\pi \eta_0(v_0)] dv_0$

but from (95.84)

$$\int [4\pi \eta_0 - c \chi_0 E_0] dv_0 = \frac{DE_0}{Dt} - E_0 \frac{D \ln \rho_0}{Dt} + 4\pi \rho_0 \frac{\partial}{\partial r} (r^2 F_0) - \rho_0 \frac{D \ln \rho_0}{Dt} - \frac{v}{r} (3P_0 - E_0) + \frac{2a F_0}{c^2}$$

$$= \rho_0 \frac{D}{Dt} \left(\frac{E_0}{\rho_0} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_0) + \rho_0 \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) - \frac{v}{r} (3P_0 - E_0) + \frac{2a F_0}{c^2}$$

$$= \rho_0 \left\{ \frac{D}{Dt} \left(\frac{E_0}{\rho_0} \right) + \rho_0 \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) - \frac{v}{\rho_0 r} (3P_0 - E_0) \right\} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_0) + \frac{2a F_0}{c^2}$$

Drop the $\frac{a}{c^2}$ term; and substitute into gas energy eqn:

$$\rho_0 \left[\frac{D\epsilon}{Dt} + p \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) \right] = \rho_0 \epsilon - \rho_0 \left\{ \frac{D}{Dt} \left(\frac{E_0}{\rho_0} \right) + \rho_0 \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) - \frac{v}{\rho_0 r} (3P_0 - E_0) \right\} - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_0)$$

$$\Rightarrow \rho_0 \left[\frac{D}{Dt} \left(\epsilon + \frac{E_0}{\rho_0} \right) + (p + P_0) \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) - \frac{v}{\rho_0 r} (3P_0 - E_0) \right] = \rho_0 \epsilon - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_0)$$

further

$$\frac{1}{\rho_0 r^2} \frac{\partial}{\partial r} (r^2 F_0) = \frac{1}{4\pi \rho_0 r^2} \frac{\partial}{\partial r} (4\pi r^2 F_0) = \frac{\partial}{\partial r} (4\pi r^2 F_0)$$

hence

$$(96.9) \quad \frac{D}{Dt} \left(\epsilon + \frac{E_0}{\rho_0} \right) + (p + P_0) \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) - \frac{v}{\rho_0 r} (3P_0 - E_0) = \epsilon - \frac{\partial}{\partial r} (4\pi r^2 F_0)$$

for isotropic radiation (diffusion), $3P_0 - E_0 = 0$, $L_r = 4\pi r^2 F_0$

$$(96.10) \quad \text{then } \frac{D}{Dt} \left(\epsilon + \frac{E_0}{\rho_0} \right) + (p + P_0) \frac{D}{Dt} \left(\frac{1}{\rho_0} \right) = \epsilon - \frac{\partial L_r}{\partial r}$$

Mechanical Energy Equation: take v. (96.2):

$$v \left\{ \rho_0 \frac{Dv}{Dt} \right\} = v \left\{ -\frac{GM_r \rho_0}{r^2} - \frac{\partial p}{\partial r} + \frac{1}{c} \int \chi_0(v_0) F_0(v_0) dv_0 \right\}$$

$$\Rightarrow \rho_0 \frac{D}{Dt} \left(\frac{1}{2} v^2 \right) = -\frac{GM_r \rho_0 v}{r^2} - v \frac{\partial p}{\partial r} + \frac{v}{c} \int \chi_0 F_0 dv_0$$

÷ by ρ_0 : $\frac{D}{Dt} \left(\frac{v^2}{2} \right) = -\frac{GM_r v}{r^2} - \frac{v}{\rho_0} \frac{\partial p}{\partial r} + \frac{v}{c \rho_0} \int \chi_0 F_0 dv_0$

$$\text{now } v = \frac{Dr}{Dt} \text{ and } \frac{v}{\rho_0} \frac{\partial p}{\partial r} = \frac{1}{\rho_0} \frac{\partial (pv)}{\partial r} - \frac{p}{\rho_0} \frac{\partial v}{\partial r} = \frac{1}{4\pi \rho_0} \frac{\partial}{\partial r} (4\pi \rho_0 v) - \frac{p}{\rho_0} \frac{\partial v}{\partial r}$$

$$\text{So } \frac{D}{Dt} \left(\frac{V^2}{2} \right) + \frac{GM_r}{r^2} \frac{Dr}{Dt} + \frac{1}{4\pi\rho_0} \frac{\partial}{\partial r} (4\pi\rho v) - \frac{P}{\rho_0} \frac{\partial v}{\partial r} = \frac{v}{c\rho_0} \int \rho_0 E_0 dv_0$$

$$= \frac{D}{Dt} \left(\frac{V^2}{2} \right) - GM_r \frac{D}{Dt} \left(\frac{1}{r} \right) + \frac{r^2 \partial}{\partial M_r} (4\pi\rho v) - \frac{P}{\rho_0} \frac{\partial v}{\partial r} = \frac{v}{c\rho_0} \int \rho_0 E_0 dv_0$$