SINGLE STAGE NUCLEAR ROCKET STUDY

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SINGLE STAGE NUCLEAR ROCKET STUDY

(Title Unclassified)

by

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ABSTRACT

This report describes a study which was carried out primarily on ground-launched, single-stage, hydrogen-propelled nuclear rockets. The missions considered were flights into circular orbits and ballistic flights.

Particular emphasis was given to determine the effect of missile trajectory, for a specified mission, on payload. The types of trajectories considered were the powered-all-the-way (PAW) ascent, the ballistic ascent, and the elliptic ascent. In the PAW missions, the missile was flown from the ground, along a trajectory that maximized payload, into a prescribed circular orbit. The PAW results were then compared, for a particular mission, with ballistic and elliptic ascent calculations. The results show that most of the missions studied were very sensitive to the type of trajectory flown, thus clearly demonstrating the gains to be made by using a nuclear rocket motor which has a recycle capability.
Mission performance results have been obtained for various single-stage nuclear systems, but with emphasis on missions which require large payloads in orbit. Present day technology indicates that a nuclear system having a launch weight of ~400,000 lb would be capable of placing a payload in excess of 40,000 lb into a 300-mi circular orbit.
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I. Introduction

The purpose of this report is to describe a study which was carried out primarily on ground-launch, single-stage, hydrogen-propelled nuclear rockets. The performance of a nuclear rocket containing a fuel made up of hydrogen and methane (3 mole %) as well as a brief discussion of a vehicle employing tank staging are also included in this report. The missions considered in this study were flights into circular orbits and ballistic flights. Sections I and II are a review and extension of the work presented by K. W. Ford. The emphasis in Ref. 1 was to fly a given payload a maximum distance. The emphasis here is to fly a rocket from the ground, along a trajectory that maximizes payload, into a prescribed circular orbit. The types of trajectories considered were the powered-all-the-way (PAW) ascent, the ballistic ascent, and the elliptic ascent. In the PAW ascent the vehicle engine is operated continuously until the desired orbit is achieved. In the ballistic ascent the vehicle is carried to a summit (by means of a propulsion and free-flight phase) which lies near the desired orbit. At the summit, the vehicle must, in a second propulsion period, produce the velocity increment required for transfer into the desired orbit. In the well-known Hohmann transfer ellipse (elliptic ascent), the vehicle
first attains circular velocity at a relatively low altitude (taken to be 100 statute miles in this report). Subsequently, the vehicle is accelerated to the perigee velocity of the transfer ellipse, at whose apogee an additional short burst of power is required for entering the satellite orbit.

In this work, the philosophy has been to consider in detail the aspects of a particular mission problem only if it is justified by present day knowledge. For this reason, an accurate formulation of the equations for air drag, air pressure, trajectory, and trajectory integration, etc., are used, whereas the equations that pertain to rocket component weights have been simplified as far as possible. The question of the type and weight of the rocket motor, when the reactor weight is not available from a detailed engineering study, has been avoided in a given mission problem by lumping together the reactor and payload weight. As more data become available, the estimates of tank, structure, and other weights given here will change, thereby necessitating changes in the results given in this report.

The following few paragraphs will describe the formulation used to solve a given mission problem in a qualitative way. Details of the formulation are given
in sections II to VI. Conclusions are given in section VII.

Section II, Staging of the Rocket on the Ground, is concerned with the determination of component masses and sizes and propellant flow rate. In section II the takeoff weight is written as the sum of 10 component weights:

1. $w_L$, payload plus nose cone plus guidance, given
2. $w_{H2}^F$, hydrogen propellant weight
3. $w_{CH4}^F$, methane propellant weight
4. $w_{H2}^T$, hydrogen tank weight, calculated from hydrogen propellant weight
5. $w_{CH4}^T$, methane tank weight, calculated from methane propellant weight
6. $w_{H2}^S$, hydrogen structure weight, given as a fixed fraction of $w_{H2}^T$
7. $w_{CH4}^S$, methane structure weight, given as a fixed fraction of $w_{CH4}^T$
8. $w_P^T$, weight of turbopumps and plumbing, written as a function of pump discharge pressure and volume flow rate
9. \( W_R \), weight of reactor plus pressure shell plus gimbals, given

10. \( W_N \), weight of the nozzle, written as a function of nozzle throat area.

The exhaust velocity can be calculated from the chamber temperature and pressure. From this, the mass flow rate and total mass can be calculated which will give the specified initial acceleration.

Section III is concerned with the trajectory calculation through the atmosphere. The trajectory is vertical for a time \( t_k \) (kickover time), at which time the thrust angle to the vertical is changed to \( \beta_k \) (kickover angle). From the time \( t_k \) until the rocket passes through the sensible atmosphere (height \( h_2 \)), the rocket flies along a gravity-turn trajectory (thrust vector parallel to velocity). The effects of air pressure and air density are included in this calculation and are obtained from an exponential approximation. Mach number and drag coefficient are calculated as functions of altitude from the equations in section III B of this report.\(^3\) The effect of air pressure on thrust and the variation of the gravitational force with altitude are taken into account.

In section IV the rocket is powered all the way (PAW) from the height \( h_2 \), along a trajectory that maximizes
payload, into a prescribed circular orbit. For this part of the trajectory the magnitude of the thrust as well as the mass flow rate were kept constant. The mathematical formulation used in section IV follows closely the work of L. Turner on minimum time of flight trajectories.4

In section V the rocket is flown from the height $h_2$, along a trajectory that maximizes payload, into a 100-mi circular orbit. The rocket is flown from the 100-mi orbit to any other specified circular orbit by means of a Hohmann transfer ellipse. Section VI gives a brief discussion of the ballistic ascent trajectory, and conclusions are given in section VII.

II. Staging of the Rocket on the Ground

A. Weights of Components

1. Payload weight, $W_L$, given (includes nose cone and guidance) in lb

2. Propellant weight (hydrogen), $W_{H_2}^F$, given in lb

3. Propellant weight (methane), $W_{CH_4}^F$, given in lb

4. Hydrogen tank weight,

$$w_{H_2}^T = \frac{Q_1(K + 2)}{(1 - a)(4/3 + K)} \left( \frac{\rho_T}{\rho_{H_2}} \right) (1 - F)W_F^T$$
5. Methane tank weight,

\[ W_{\text{CH}_4} = \frac{Q_2(K + 2)}{(1 - a)(4/3 + K)} \left( \frac{\rho_T}{\rho_{\text{CH}_4}} \right)(F) W_F^T \]

The equations for the tank weights were derived on the assumption of a particular tank configuration. The configuration chosen was a tank of radius \( R_T \), with the cylindrical straight section of length \( \ell_T = K R_T \), and hemispherical endcaps. The over-all length of the rocket is \( L = K R_T + 2R_T \). The tank walls are made of steel of constant thickness. In the foregoing equations,

\( \rho_T \) = steel density in \( \text{lb/ft}^3 \) (value used was 494)
\( \rho_{\text{H}_2} \) = hydrogen density in \( \text{lb/ft}^3 \) (value used was 4.43)
\( \rho_{\text{CH}_4} \) = methane density in \( \text{lb/ft}^3 \) (value used was 43.68)
\( F \) = weight percent of methane
\( W_F^T \) = total fuel weight (hydrogen plus methane)
\( a \) = fraction of tank volume left for air space

(ullage factor)

\[ Q_1 = 2(f_s P_T^{\text{H}_2})/\sigma \]
\[ Q_2 = 2(f_s P_T^{\text{CH}_4})/\sigma \]

where

\( f_s \) = factor of safety
\( P^{H_2}_T \) = hydrogen tank pressure in psi (gas overpressure plus hydrostatic head)

\( P^{CH_4}_T \) = methane tank pressure in psi (gas overpressure plus hydrostatic head)

\( \sigma \) = yield strength in psi

In this work \( K = 10, a = 0.03, f_s = 1.39 \) and was taken from LA-1870, and \( \sigma = 150,000 \). The value used for the overpressure in the tanks was 25 psi (based on the Atlas design), the value of the hydrogen hydrostatic head was assumed to be 10 psi, and the value of the methane hydrostatic head was assumed to be 35 psi.

6. Hydrogen structure weight assumed is

\[ W^{H_2}_S = 0.15 P^{H_2}_T \text{ in lb} \]

7. Methane structure weight assumed is

\[ W^{CH_4}_S = 0.15 P^{CH_4}_T \text{ in lb} \]

8. The total pump weight \( (W^T_P) \) is

\[ W^T_P = W^T_{P_1} + W^T_{P_2} \text{ in lb} \]

Since the present state of hydrogen pump development is in its infancy and no methane pump development is in progress, no attempt was made to estimate separate pump weights. What was done was to assume that
\[ W_p^T \approx W_p^{H_2} = \frac{C_p(1 - F) W_F^T (P_d)^{1/2}}{\rho_{H_2}} \] (1)

since only problems containing small admixtures of methane were considered in this report. In Eq. '1

\[ W_F^T = \text{total propellant flow rate in lb/sec} \]

\[ \rho_{H_2} = \text{hydrogen propellant density} \]

\[ P_d = \text{pump discharge pressure in lb/in.}^2 \]

and

\[ C_p = 1.28 \] (see page 287 of Ref. 3)

9. Reactor weight plus pressure shell plus reflector.

No attempt was made to calculate reactor weights when detailed engineering data were unavailable. When this was the case, the calculations were performed by lumping together the reactor and payload weights.

10. The nozzle weight is

\[ W_N = C_N A_t \]

where

\[ A_t = \text{throat area in in.}^2 \]

\[ C_N = 6.8 \] (the value used for the nozzle coefficient is thought to be conservative)
B. Mass Flow Rate, Thrust, Takeoff Weight*

1. Exit pressure. For a gamma law gas and adiabatic expansion, the ratio of exit area to throat area is related to the ratio of exit pressure to chamber pressure by (see Ref. 7, page 61)

\[
\frac{A_e}{A_t} = \left( \frac{2}{k+1} \right)^{(k-1)/k} \left( \frac{p_e}{p_c} \right)^{-1} \left\{ \frac{k+1}{k-1} \left[ 1 - \left( \frac{p_e}{p_c} \right)^{(k-1)/k} \right] \right\}^{-1/2}
\]  

(2)

where \( k \) is the ratio of the specific heat at constant pressure to the specific heat at constant volume. In this problem the area ratio is specified, as is the chamber pressure \( p_c \). Hence this equation yields a value for the exit pressure \( p_e \).

2. Exhaust velocity. The exhaust velocity for a perfect, infinitely expanding nozzle is

\[
v_e^0 = \left( \frac{2k}{k-1} \right) \left( \frac{g_0 R T_c}{M_{av}} \right)^{-1/2}
\]

(3)

where

- \( R \) = the universal gas constant
- \( R = 2782 \text{ ft-lb/lb-mole } ^\circ\text{K} \)
- \( T_c \) = the chamber temperature in \(^\circ\text{K}\)

*Portions of section B were taken from an internal memorandum by K. W. Ford, a LASL consultant.
\( g_0 = \text{a constant} = 32.17 \text{ ft/sec}^2 \)

\( M_{av} = \text{the average molecular weight, defined by} \)

\[
M_{av} = \frac{\sum \bar{n}_i M_i}{\sum \bar{n}_i}
\]

where

\( \bar{n}_i = \text{mole percent of constituent} \ i \)

\( M_i = \text{molecular weight of constituent} \ i \)

The actual exhaust velocity is

\[
v_e = v_{e0} C_{NL} \left[ 1 - \left( \frac{p_e}{p_c} \right)^{(k-1)/k} \right]^{1/2}
\]

where \( C_{NL} \) is a nozzle efficiency factor, set equal to 0.97 for the problems discussed in this report.

3. Effective exhaust velocity and specific impulse.

To the exhaust velocity must be added a term due to finite exit pressure and finite air pressure, which we call the "pressure velocity,"

\[
v_p = \left( 1 - \frac{p_a}{p_e} \right) C^* \left( \frac{A_e}{A_t} \right) \left( \frac{p_e}{p_c} \right)
\]

where \( p_a \) is air pressure and \( C^*, \) the characteristic velocity, is given by
The specific impulse is

$$I_{sp} = \frac{(V_e + V_p)}{g_0}$$  \hspace{1cm} (8)

where $V_e + V_p$ is the effective exhaust velocity. Because of the flow separation in the nozzle, the ratio $P_a/P_e$ cannot become arbitrarily large. In this problem we limit the ratio to 3. If $P_e$ is calculated to be less than $P_a/3$, we arbitrarily replace $(1 - P_a/P_e)$ in Eq. 6 by -2.

4. **Mass flow rate.** The effective exhaust velocity and given desired initial takeoff acceleration, $g_0a_0$, specify the mass flow rate,

$$\dot{m}_F = \frac{g_0W_0(1 + a_0)}{(1 - \beta)(V_e + V_p)} = \frac{W_0(1 + a_0)}{(1 - \beta)(I_{sp})}$$  \hspace{1cm} (9)

where $\beta$ is the fraction of the propellant used to drive the pump. This fraction is assumed to be exhausted at negligible velocity.

5. **Throat area.** The throat area is given by

$$A_t = \frac{C* \dot{m}_F}{(P_c'g_0)}$$  \hspace{1cm} (10)
6. **Reactor power.** The reactor power in Mev is given by

\[ P_R = C_R \frac{\dot{W}}{F} \Delta T \]  

(11)

where \( \Delta T \) = the temperature drop of the gas through the reactor, in °C

\[ C_R = 7.21 \times 10^{-3} \text{ megajoules/lb °C} \]

The alteration of specific heat due to admixed CH\(_4\) and the question of dissociation and recombination are not taken into account.

7. **Launch weight equation.** An explicit formula for total takeoff mass \( W_0 \) may be arrived at from the formulas given above, and is

\[
W_0 = \frac{W_L + W_R + W_T}{(K+2)Q_1 \rho_T} \left[ \frac{1.15 \left( \frac{Q_2}{Q_1} \rho_{H_2} - \rho_{CH_4} \right) F + 1.15 \rho_{CH_4}}{\rho_{H_2} \rho_{CH_4}} \right] + 1
\]

(12)

\[
1 - \frac{g_0 (1+a_0)}{(1-g) (V_e + V_p)} \left[ (1-F) C_p P d^{1/2} + \frac{C_n C^*}{(P_c g_0)} \right]
\]

Once \( W_0 \) is determined from Eq. (12), all component weights can be found as well as the mass flow rate \( \dot{W}_F \). The total burning time, if all the fuel is used, is

\[ t_B = \frac{W_T (1 - a)}{\dot{W}_F} \]  

(13)
where \( \alpha \) is the fraction of unused propellant at burnout (holdover).

With a slight modification of Eq. 12, the equations given in this section can be used to determine the performance of a chemical system. To do this, Eq. 1 must be changed so that \( W_P^T \) is the sum of two pump weights, which in turn introduces an additional term in Eq. 12.

III. Trajectory Calculation through the Atmosphere for a Non-Rotating Circular Earth

For this part of the flight the rocket is flown vertically for a time \( t_k \) (kickover time), at which time the thrust angle to the vertical is changed to \( \beta_k \) (kickover angle). From the time \( t_k \) until the rocket passes through the sensible atmosphere, the rocket flies along a gravity-turn trajectory (thrust vector parallel to velocity). The equations of motion for this part of the flight (see Fig. 1) are

\[
\dot{V} = \frac{(T - D)g_0}{W} - g \cos\beta
\]

\[
\dot{\beta} = -\left(\frac{V \sin\beta}{R} - \frac{g \sin\beta}{V}\right)
\]

\[
\dot{R} = V \cos\beta
\]

\[
\dot{\theta} = \frac{V}{R} \sin\beta
\]
Fig. 1 Coordinate system for gravity-turn portion of powered flight
where

\( V = \) absolute value of the tangential velocity

\( \beta = \) angle the velocity vector makes with local vertical

\( R = \) distance from center of earth to missile

\( \theta = \) angle the radius vector makes with \( y \) axis

\( T = \) rocket thrust in pounds

\( D = \) drag force in pounds

In order to solve this set of simultaneous, nonlinear, first order differential equations we need to obtain \( Tg_0/W, \) \( Dg_0/W, \) and \( g \) in terms of the variables of integration. These quantities will now be evaluated.

A. Evaluation of \( Tg_0/W \)

The specific thrust (\( Tg_0/W \)) is a function of \( W(t) \) and changing air pressure. The effect of \( W(t) \) is to cause the specific thrust to be proportional to \( \left[ 1 - \left( \frac{W}{W_0} \right) T \right]^{-1}, \)

and the effect of changing air pressure causes the specific thrust to be proportional to \( (V_e + V_p)/(V_e + V'_p) \). The value of \( V_p \) is

\[
V_p = \frac{V'_p (1 - P_a/P_e)}{(1 - 14.7/P_e)} \tag{18}
\]

except that the denominator and/or the quantity in parentheses in the numerator are arbitrarily replaced by \(-2\) if either is calculated to be less than \(-2\). Thus \( Tg_0/W \)
can be written as

\[
\frac{Tg_0}{W} = \left( \frac{Tg_0}{W} \right)_0 \left[ \frac{(V_e + V_p)}{(V_e + V_p^0)} \right] \left[ 1 - \left( \frac{W_f}{W_0} \right) t \right]^{-1}
\]  

(19)

where \(\left( \frac{Tg_0}{W} \right)_0\) is the initial specific thrust and is \(g_0(1 + a_0)\). The value of \(P_a\) was obtained from the equation

\[
P_a = 14.7 e^{-\left( h/22,800 \right)}
\]  

(20)

where \(h\) is the altitude in feet (see Fig. 1).

B. Evaluation of \(Dg_0/W\)

The equation for the specific drag (the value of \(D\) was obtained from page 373 of Ref. 7) is

\[
\frac{Dg_0}{W(t)} = \frac{\pi g_0 f \rho_a(h) V^2}{8W(t)}
\]  

(21)

where

- \(D_T\) = tank diameter in ft
- \(\rho_a\) = air density in slugs/ft\(^3\)
- \(g_0\) = 32.2 ft/sec\(^2\)
- \(V\) = speed in ft/sec
- \(W(t) = W_0 - \dot{W}_f t\)

The value of \(\rho_a\) was obtained from the equation

\[
\rho_a(h) = AT e^{-\left( h/22,800 \right)}
\]  

(22)
where \( \Delta T = 0.003 \) and \( h \) is in feet. The formulas used to obtain the drag coefficient are

\[
\begin{align*}
C_D &= 0.37 \quad \text{if} \quad M < 0.9 \\
C_D &= 0.60 + 2.3(M - 1) \quad \text{if} \quad 0.9 \leq M \leq 1.1 \\
C_D &= 0.30 + (0.60/M) \quad \text{if} \quad M > 1.1
\end{align*}
\]

where \( M = V/c = \text{Mach number} \). Sound speed, \( c \), is represented by

\[
\begin{align*}
c &= 1120 - 0.00414h \quad \text{if} \quad h < 35,000 \text{ ft} \\
c &= 975 \text{ ft/sec} \quad \text{if} \quad h \geq 35,000 \text{ ft}
\end{align*}
\]

The value of \( c \) should increase again at very high altitude, but by this time, the drag is small, and an accurate value of \( c \) is not important. It is felt that the relative slope of the curve represented by these equations is correct, but the absolute values of the numbers to be assigned to the ordinate are in doubt. The coefficient \( f_D \) is introduced into the set of input parameters in order to facilitate easy changes in \( C_D \) as better information on \( C_D \) vs \( M \) becomes available. (For the problems in this report \( f_D \) was taken to be 1.)

**C. Gravity Term**

The gravitational acceleration is

\[
g = g_0 \left( \frac{R_e}{R_e + h} \right)^2 \tag{23}
\]
where $R_e$, the radius of the earth, is $2.0908 \times 10^7$ ft.

1. **Solution of the equations.** Having $Tg_0/W$, $Dg_0/W$, and $g$ in terms of the variables of integration, values of $\theta$, $V$, $R$, and $\theta$ were obtained as a function of time by numerically solving, by means of Runge-Kutta, Eqs. 1 through 5. The calculations were carried out on an IBM 704 computer. The program for this problem is given in the appendix.

A possible flight demonstration mission to prove the possibility of nuclear rocket propulsion would be to lob a vehicle to an altitude of ~500 to ~1000 mi and then let it impact a few hundred miles from the launch pad. A reactor operating at a temperature of 2000°C and running at a power level of ~1000 Mw, Kiwi C design, has the ability to generate enough thrust for this mission. Using the equations developed in section II, the launch weight of such a vehicle was found to be ~41,000 lb. The vehicle component weight breakdown is given in Table 1. The trajectory history, obtained from the equations developed in this section, is given in Fig. 2. The input parameters used in working this problem, and which are common to all problems discussed in the report, are also given in Table 1.
TABLE 1

Mission "Lob Shot" Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Launch weight</td>
<td>40,600 lb</td>
</tr>
<tr>
<td>Flow rate</td>
<td>79.3 lb/sec</td>
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<tr>
<td>Reactor power</td>
<td>1140 Mw</td>
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<tr>
<td>Reactor exit gas temp</td>
<td>2000°C</td>
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<tr>
<td>Pump and turbine weight</td>
<td>1000 lb</td>
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<tr>
<td>Nozzle weight</td>
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<td>Structure weight</td>
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<td>Tank weight</td>
<td>2100 lb</td>
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<td>Fuel weight</td>
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<td>Holdover weight</td>
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<tr>
<td>Tank diameter</td>
<td>11.3 ft</td>
</tr>
<tr>
<td>Ratio of over-all length to diameter</td>
<td>6</td>
</tr>
<tr>
<td>Velocity at burnout</td>
<td>14,050 ft/sec</td>
</tr>
<tr>
<td>Maximum height</td>
<td>1050 mi</td>
</tr>
<tr>
<td>Total burning time</td>
<td>341 sec</td>
</tr>
<tr>
<td>Distance from launch pad to impact</td>
<td>300 mi</td>
</tr>
<tr>
<td>Payload</td>
<td>100 lb</td>
</tr>
<tr>
<td>Total time in air</td>
<td>1685 sec</td>
</tr>
<tr>
<td>Holdover</td>
<td>0.01</td>
</tr>
<tr>
<td>Ullage</td>
<td>0.03</td>
</tr>
<tr>
<td>Ratio of length to diameter</td>
<td>6</td>
</tr>
<tr>
<td>Pump discharge pressure</td>
<td>1500 psi</td>
</tr>
<tr>
<td>Chamber pressure</td>
<td>1070 psi</td>
</tr>
<tr>
<td>Fuel fraction used to drive pump</td>
<td>0.05</td>
</tr>
<tr>
<td>Takeoff acceleration</td>
<td>0.35 g₀</td>
</tr>
<tr>
<td>Effective k of hot hydrogen</td>
<td>1.35</td>
</tr>
<tr>
<td>Nozzle exit area to throat area</td>
<td>25</td>
</tr>
<tr>
<td>Nozzle eff factor</td>
<td>0.97</td>
</tr>
</tbody>
</table>
Fig. 2 Trajectory history obtained from gravity-turn equations
IV. Powered-All-the-Way Trajectory Optimization

A. Derivation of Optimization Equations

The direction of the thrust is programmed in such a way as to minimize the time required for the rocket to fly from the top of the atmosphere \((h_2)\) to the prescribed orbit. During this portion of the flight both \(W\) and \(I_{sp}\) are kept constant. Since both \(W\) and \(I_{sp}\) are constant, minimum flight time means maximum payload. The required mathematical formulation will now be derived.

Let \(R\) and \(\Theta\) be polar coordinates of a point in space with respect to an inertial coordinate system whose origin is at the center of the earth (see Fig. 3). In Fig. 3, \(\beta\) is the angle the velocity vector makes with the local vertical, \(\gamma\) the angle the thrust vector makes with the local vertical, and \(h\) the altitude of the rocket above the surface of the earth. At time \(t_2\) a missile with mass \(W_{02}\) is at \((h_2, \beta_2)\) and has velocity \(V_2\). It is flown from \(h_2\) and arrives at the prescribed circular orbit after a time \(t' = t_3 - t_2\), which is unknown. For this portion of the flight the time scale was shifted an amount \(t_2\) for convenience in the integrations. To obtain the actual flight time, \(t_2\) must be added to \(t'\). It is desired to find functions \(\eta_R\) and \(\eta_{\Theta}\), the specific thrusts in the direction of \(R\) and \(\Theta\) (see Fig. 3), which minimize \(t'\) under
Fig. 3 Coordinate system used for trajectory optimization portion of flight
the given constraints. The functions $\eta_R$ and $\eta_\theta$ satisfy

$$\eta_R^2 + \eta_\theta^2 = \left(\frac{\Omega_0 W_0}{W_0 - \dot{W}_t}\right)^2$$

(24)

from $t = 0$ to $t = t'$, after which they are both zero. The value of $t'$ can be determined for a specific orbit altitude

from a knowledge of $\eta_R$ and $\eta_\theta$.

The equations of motion are

$$\ddot{R} - R\ddot{\theta} + \frac{\kappa}{R^2} = \eta_R$$

(25)

and

$$2R\ddot{\theta} + \dot{R}\ddot{\theta} = \eta_\theta$$

(26)

where

$$\kappa = g_0 R_e^2$$

Letting $U = \dot{R}$ and $L = R^2 \dot{\theta}$ leads to the following set of equations:

$$\frac{\dot{L}}{R} - \eta_\theta = 0 = \psi_1$$

(27)

$$\dot{U} - \frac{L^2}{R^3} + \frac{\kappa}{R^2} - \eta_R = 0 = \psi_2$$

(28)

$$\dot{R} - U = 0 = \psi_3$$

(29)

plus the equation

$$\eta_R^2 + \eta_\theta^2 - \left(\frac{\Omega_0 W_0}{W_0 - \dot{W}_t}\right)^2 = 0 = \psi_4$$

(30)
Now we wish to fly the missile from height $h_2$ to height $h_3$ in the minimum amount of time, thus expending the minimum amount of fuel. To do this we minimize the time subject to the above differential constraints and the initial conditions $R = R_2$, $U = U_2$, $L = L_2$ and the final (or end) conditions $R = R_3$, $U = U_3$, and $L = L_3$, where $L_3$ is the angular momentum necessary to have a circular orbit of radius $R_3$. The integral to be minimized is

$$I = \int_{t=0}^{t=t'} F(R, U, L, \eta_R, \eta_Q, U, L, R, t) \, dt \quad (31)$$

where

$$F = 1 + \sum_{i=1}^{4} \lambda_i(t) \psi_i, \quad i = 1 \text{ to } 4 \quad (32)$$

Now in order that $\delta I = 0$, the function $F$ must satisfy the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{y}_i} \right) - \frac{\partial F}{\partial y_i} = 0 \quad (33)$$

and in addition the transversality condition $^8$ at $t'$. This condition is

$$\left[ (F - \sum_{i=1}^{4} y_i \frac{\partial F}{\partial \dot{y}_i}) \, dt + \sum_{i=1}^{4} \frac{\partial F}{\partial \dot{y}_i} \, dy_i \right]_{t'} = 0 \quad (34)$$

where $dt$ and $dy_i$ pertain to the final curve (circular orbit) and the coefficients of $dt$ and $dy_i$ pertain to the
path leading into the final curve. Substituting $F$ into Eq. 33 leads to the following set of equations:

$$\begin{align*}
\dot{\lambda}_3 &= \dot{\lambda}_1 \frac{L}{R^2} - \lambda_2 \left(3\frac{L^2}{R^4} - \frac{2R}{R^3}\right) = 0 \\
\dot{\lambda}_1 &= \lambda_1 \frac{U}{R} + 2\lambda_2 \frac{L}{R^2} = 0 \\
\dot{\lambda}_2 + \lambda_3 &= 0 \\
\lambda_2 - 2\eta_R \lambda_4 &= 0 \\
\lambda_1 - 2\eta_0 \lambda_4 &= 0
\end{align*}$$

(35) (36) (37) (38) (39)

Eliminating $\lambda_4$ from Eqs. 38 and 39 and making use of Eq. 30 yields

$$\eta_R = \frac{\lambda_2}{\left(\lambda_2^2 + \lambda_1^2\right)^{1/2}} \left[ \frac{g_0 W_{Isp}}{\left(W_{02} - \dot{W}_t\right)} \right]$$

(40)

and

$$\eta_0 = \frac{\lambda_1}{\left(\lambda_2^2 + \lambda_1^2\right)^{1/2}} \left[ \frac{g_0 W_{Isp}}{\left(W_{02} - \dot{W}_t\right)} \right]$$

(41)

Equations 40 and 41 show that $\lambda_2$ is a direction number along the radius vector and that $\lambda_1$ is a direction number normal to the radius vector. Substituting $F$ into Eq. 34 and writing the equation of the orbit as a function
of $t$, namely, $R = C$ (a constant), $L = C$, and $U = 0$, yields
\[
\left\{ \frac{F}{R} + \left( \frac{L\lambda_1}{R} \right) + \dot{U} + \dot{R} \lambda_3 \right\} dt \bigg|_{t'=0} = 0 \tag{42}
\]
and since $\Psi_i = 0$, $\dot{R}_3 = 0$, Eq. 42 becomes
\[
\left( \frac{L\lambda_1}{R} + \dot{U} \right) \bigg|_{t'=0} = 1 \tag{43}
\]
Since $\dot{U} = \eta \dot{R}$ and $\dot{L} = R \eta \dot{\theta}$ at $t = t'$, Eq. 43 can be written as
\[
\left( \eta \lambda_1 + \eta R \lambda_2 \right) \bigg|_{t'=0} = 1 \tag{44}
\]
which is our transversality condition.

From Eqs. 40, 41, and 44 the equation for the final time ($t'$) in terms of the directional components of the thrust is
\[
t' = \left( \frac{\omega_0}{2} \right) - \frac{\eta g_0}{I_{sp}} \left( \lambda_1^2 + \lambda_2^2 \right)^{1/2} \tag{45}
\]
and since $t_2$ is known, the actual flight time to enter into the prescribed circular orbit is $t_3$, where $t_3 = t' + t_2$.

For convenience we now group the equations to be solved together. They are
\[
\begin{align*}
\dot{L} - R \eta \theta &= 0 \\
\dot{U} - \frac{L^2}{R^3} + \frac{K}{R^2} - \eta \dot{R} &= 0 \\
\dot{R} - U &= 0
\end{align*}
\]

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\begin{align}
\dot{\lambda}_3 + \lambda_1 \frac{\eta Q}{R} - \lambda_2 \frac{3L^2}{R^4} - \frac{2k}{R^3} &= 0 \\
\dot{\lambda}_1 - \frac{U}{R} + 2L^2 &= 0 \\
\dot{\lambda}_2 + \lambda_3 &= 0
\end{align}

The boundary conditions at \( t = 0 \) are \( R = R_2, \ L = L_2, \ U = U_2 \) and are given in section III (obtained from the solution of the gravity-turn equations). The boundary conditions at \( t = t' \) are \( R = R_3, \ U = 0, \ L = L_3 = (kR_3)^{1/2} \).

In the above set of equations we wish to solve for \( L, \ U, \ R, \ \lambda_1, \ \lambda_2, \ \lambda_3 \) as functions of time.

**B. Numerical Solution of the Boundary Value Problem**

To obtain a numerical solution to the above boundary value problem, a suitable first guess of \( \lambda_1, \ \lambda_2, \ \lambda_3 \) at \( t = t' \) must be made. Examination of results found with an IBM 704 computer has shown that a suitable first guess for the \( \lambda \)'s is

\[
\lambda_1 = \frac{W_{02}}{2.8 \dot{W} (V_e + V_p)} = \frac{W_0 - \dot{W}_F t_B}{2.8 \dot{W} (V_e + V_p)}
\]

\[
\lambda_2 = - \frac{1}{2} \lambda_1^0
\]

\[
\lambda_3 = \frac{0.71}{V_e + V_p}
\]
If this leads to a negative $t'$, the machine program automatically selects another initial set of $\lambda$'s from among a group of five "built-in" guesses. If all of these fail, the machine prints out "$T$ is negative" and the problem is terminated at this stage of the calculation.

To clarify notation the $\lambda$ guesses will have a numbered superscript starting with zero; the first guess of $\lambda_1$, $\lambda_2$, $\lambda_3$ at $t = t'$ is $\lambda_1^0$, $\lambda_2^0$, $\lambda_3^0$.

Using the first guess for the $\lambda$'s ($\lambda_1^0$, $\lambda_2^0$, $\lambda_3^0$) along with $L_3$, $R_3$, and $U_3$, start at $t'$ and integrate back through the set of differential equations, by means of Runge-Kutta, to obtain $R(0) = R^0$, $U(0) = U^0$, and $L(0) = L^0$. Let $\lambda_{11}$, $\lambda_{21}$, and $\lambda_{31}$ be the final values of these functions in the correct solution. The correct values of $R$, $U$, and $L$ at $t = 0$ are $R(0) = R_0 = R_u$, $U(0) = U_0 = U_2$ and $L(0) = L_0 = L_2$.

A systematic method for generating additional sets of $\lambda$'s such that

$$R^i - R_0 \leq \varepsilon_R$$
$$U^i - U_0 \leq \varepsilon_U$$
$$L^i - L_0 \leq \varepsilon_L$$

will now be given. In these equations $i$ represents the $i$th set of $\lambda$'s used and $\varepsilon_R$, $\varepsilon_U$, and $\varepsilon_L$ are prescribed.
errors in radius, radial velocity, and angular velocity.

We now relate the \( \lambda' \)'s at \( t = t' \) to the values of \( L, U, \) and \( R \) at \( t = 0 \) by the equations

\[
L^0 - L_0 = f_1(\lambda_1^0, \lambda_2^0, \lambda_3^0) - f_1(\lambda_{11}, \lambda_{21}, \lambda_{31})
\]

\[
= \alpha_{11}(\lambda_1^0 - \lambda_{11}) + \alpha_{12}(\lambda_2^0 - \lambda_{21}) + \alpha_{13}(\lambda_3^0 - \lambda_{31})
\]

\[
U^0 - U_0 = f_2(\lambda_1^0, \lambda_2^0, \lambda_3^0) - f_2(\lambda_{11}, \lambda_{21}, \lambda_{31})
\]

\[
= \alpha_{21}(\lambda_1^0 - \lambda_{11}) + \alpha_{22}(\lambda_2^0 - \lambda_{21}) + \alpha_{23}(\lambda_3^0 - \lambda_{31})
\]

\[
R^0 - R_0 = f_3(\lambda_1^0, \lambda_2^0, \lambda_3^0) - f_3(\lambda_{11}, \lambda_{21}, \lambda_{31})
\]

\[
= \alpha_{31}(\lambda_1^0 - \lambda_{11}) + \alpha_{32}(\lambda_2^0 - \lambda_{21}) + \alpha_{33}(\lambda_3^0 - \lambda_{31})
\]

where the right hand side of these equations represents the first order terms of a Taylor series expansion of \( f_1(\lambda_1^0, \lambda_2^0, \lambda_3^0) \) about \( f_1(\lambda_{11}, \lambda_{21}, \lambda_{31}) \) and so forth.

Now we continue to generate sets of \( \lambda' \)'s, subject to the condition that

\[
t' = \left( \frac{W_{02}}{W} \right) - g_0 I_{sp} \left[ (\lambda_1^0)^2 + (\lambda_2^0)^2 \right]^{1/2}
\]

until

\[
f_1(\lambda_1^i, \lambda_2^i, \lambda_3^i) - f_1(\lambda_{11}, \lambda_{21}, \lambda_{31}) = 0
\]
The procedure used to satisfy Eqs. 49, 50, and 51, was the following. Equations 46, 47, 48 are first rewritten as

\[ f_2(\lambda_1^i, \lambda_2^i, \lambda_3^i) - f_2(\lambda_{11}, \lambda_{21}, \lambda_{31}) = 0 \]  
\[ f_3(\lambda_1^i, \lambda_2^i, \lambda_3^i) - f_3(\lambda_{11}, \lambda_{21}, \lambda_{31}) = 0 \]

where

\[ \lambda_1^0 - \lambda_{11} = h_1^1 \]
\[ \lambda_2^0 - \lambda_{21} = h_2^1 \]
\[ \lambda_3^0 - \lambda_{31} = h_3^1 \]

In Eqs. 52, 53, 54, the quantities \( h_1^1, h_2^1, h_3^1 \) represent a correction in the choice of the first set of \( \lambda \)'s, namely, \( \lambda_1^0, \lambda_2^0, \lambda_3^0 \). The second set of \( \lambda \)'s chosen is
\[ \lambda_1^1 = \lambda_1^0 - h_1^1 \quad (55) \]
\[ \lambda_2^1 = \lambda_2^0 - h_2^1 \quad (56) \]
\[ \lambda_3^1 = \lambda_3^0 - h_3^1 \quad (57) \]

and can be determined once \( h_1^1, h_2^1, h_3^1 \) are known. The values for \( h_1^1, h_2^1, h_3^1 \) are determined from Eqs. 52, 53, 54 once the \( \alpha \) coefficients are known. The method for determining the coefficients in Eqs. 52, 53, 54 is as follows.

Change \( \lambda_1^0 \) slightly to \( \lambda_1^0 - \Delta \lambda_1 \), and use this new set of \( \lambda \)'s, namely, \( \lambda_1^0 - \Delta \lambda_1, \lambda_2^0, \lambda_3^0 \), to obtain the time of integration \( (t') \) from Eq. 45 and then integrate through the differential equations. This yields

\[ L^0 + \Delta L - L_0 = -a_{11}\Delta \lambda_1 + a_{11}(\lambda_1^0 - \lambda_{11}) \quad (58) \]

\[ + a_{12}(\lambda_2^0 - \lambda_{21}) + a_{13}(\lambda_3^0 - \lambda_{31}) \]

\[ U^0 + \Delta U - U_0 = -a_{21}\Delta \lambda_1 + a_{21}(\lambda_1^0 - \lambda_{11}) \quad (59) \]

\[ + a_{22}(\lambda_2^0 - \lambda_{21}) + a_{23}(\lambda_3^0 - \lambda_{31}) \]

\[ R^0 + \Delta R - R_0 = -a_{31}\Delta \lambda_1 + a_{31}(\lambda_1^0 - \lambda_{11}) \quad (60) \]

\[ + a_{32}(\lambda_2^0 - \lambda_{21}) + a_{33}(\lambda_3^0 - \lambda_{31}) \]
Subtracting Eqs. 46, 47, 48 from Eqs. 58, 59, 60 yields

\[ a_{11} = - \frac{\Delta L}{\Delta \lambda_1} \]
\[ a_{21} = - \frac{\Delta U}{\Delta \lambda_1} \]
\[ a_{31} = - \frac{\Delta R}{\Delta \lambda_1} \]

The other coefficients are obtained by varying \( \lambda_2^0 \) and \( \lambda_3^0 \). The new set of \( \lambda \)'s, namely, \( \lambda_1^1, \lambda_2^1, \lambda_3^1 \), are used to obtain a new \( t' \) from Eq. 45. With this set of \( \lambda \)'s we integrate through our differential equations, starting at \( t = t' \) and integrating to \( t = 0 \), obtaining a new set of \( f \)'s which are

\[ f_1(\lambda_1^1, \lambda_2^1, \lambda_3^1) = L^1 \]
\[ f_2(\lambda_1^1, \lambda_2^1, \lambda_3^1) = U^1 \]
\[ f_3(\lambda_1^1, \lambda_2^1, \lambda_3^1) = R^1 \]

Starting with this set of \( f \)'s, the routine given above is used to obtain the new set of \( a \) coefficients. In obtaining these the same set of \( \Delta \lambda \)'s is used. Then a new set of \( h \)'s is generated, which are \( h_1^2, h_2^2, h_3^2 \). From this set of \( h \)'s, a new set of \( \lambda \)'s is constructed, which is
\[ \lambda_1^2 = \lambda_1^1 - h_1^2 \]
\[ \lambda_2^2 = \lambda_2^1 - h_2^2 \]
\[ \lambda_3^2 = \lambda_3^1 - h_3^2 \]

This procedure is continued until
\[ R_i - R_0 \leq \epsilon_R \] \hspace{1cm} (61)
\[ U_i - U_0 \leq \epsilon_U \] \hspace{1cm} (62)
\[ L_i - L_0 \leq \epsilon_L \] \hspace{1cm} (63)

where \( \epsilon_R \), \( \epsilon_U \), and \( \epsilon_L \) are prescribed errors in radius, radial velocity, and angular velocity.

At this point the optimization equations were combined with the gravity-turn equations and coded for an IBM 704 computer. Details of the code are given in the appendix.

C. Results

1. Hydrogen system. The results for single-stage missiles having launch weights of 413,000 and 72,000 lb will now be given. In these results it is assumed that, for temperatures >1500°C, one is flying a nuclear rocket which has a corroding reactor or that the reactor has graphite-loaded fuel elements which are protected with niobium carbide. (See following section.)
Figures 4 through 9 give the final results obtained for a missile having a launch weight of 413,000 lb. Figure 4 gives circular orbit altitude as a function of reactor weight plus payload with the temperature as a parameter. To convert the results given in Fig. 4 in order to obtain circular orbit altitudes as a function of payload with the temperature as a parameter requires reactor weights which can only come from a detailed engineering study. Since no detailed engineering data exist for reactors having power densities of 0.5 Mw/lb and associated temperatures in excess of 1500°C, we assume that reactor weight can be expressed as (see page 290 of Ref. 3)

\[ W_R = 10,000 + 2000 \frac{P_R}{T_C} \]  

(64)

where

- \( W_R \) = the weight of the reactor plus pressure shell
- \( P_R \) = the reactor power in Mw
- \( T_C \) = the gas temperature at the exit of the reactor in °C

Equation 64 is only used in connection with missions which have a vehicle launch weight of 413,000 lb. These missions require reactor powers which vary from 9000 Mw to 16,000 Mw and exit gas temperatures in excess of 1500°C. Equation 64
Fig. 4 Circular orbit altitude as a function of payload plus reactor weight with the temperature taken as a parameter (results for PAW trajectories)
Fig. 5 Circular orbit altitude as a function of payload with the temperature taken as a parameter (results for PAW trajectories)
Fig. 6 Reactor plus payload in a given circular orbit as a function of vacuum specific impulse with circular orbit altitude taken as a parameter (results for PAW trajectories)
Fig. 7 Payload in circular orbit as a function of vacuum specific impulse with circular orbit altitude taken as a parameter (results for PAW trajectories)
Fig. 8  Circular orbit altitude in miles as a function of mass ratio with the vacuum specific impulse taken as a parameter (results for PAW trajectories)
Fig. 9 Power density as a function of vacuum specific impulse with the circular orbit altitude taken as a parameter (results for PAW trajectories)
is used to obtain the results given in Fig. 5.

The only high power density, high temperature reactor for which a detailed engineering design exists, using hydrogen as the propellant, is the Condor reactor. The weight of a Condor reactor, operating at a temperature of 1500°C and at a power level of 9000 Mw, is 22,800 lb. This represents a power density of ~0.4 Mw/lb at a temperature of 1500°C. Substituting the Condor values for \( P_R \) and \( T_c \) into Eq. 64 yields a \( W_R \) of 22,000 lb; thus Eq. 64 is essentially anchored to the Condor results at 1500°C. Whether or not the reactor weights given by Eq. 64, for power densities greater than 0.4 Mw/lb and associated temperatures in excess of 1500°C, are possible with present day technology depends on forthcoming detailed engineering studies.

Figure 6 gives the reactor weight plus payload as a function of vacuum specific impulse with the circular orbit altitude taken as a parameter. The relationship between the gas temperature, in degrees centigrade, at the exit of the reactor and the vacuum specific impulse is given in Table 2.
TABLE 2

Effect of Temperature on Vacuum Specific Impulse

<table>
<thead>
<tr>
<th>$T_c$, °C</th>
<th>$I_{sp, vac}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>761</td>
</tr>
<tr>
<td>2170</td>
<td>807</td>
</tr>
<tr>
<td>2500</td>
<td>860</td>
</tr>
<tr>
<td>3000</td>
<td>934</td>
</tr>
<tr>
<td>3500</td>
<td>1003</td>
</tr>
</tbody>
</table>

The results given in Fig. 7 were obtained from Fig. 6 by making use of the previous relationship, Eq. 64, for reactor weight. Figure 8 gives circular orbit altitude as a function of mass ratio (launch weight/burnout weight) with the vacuum specific impulse taken as a parameter. Figure 9 is a plot of power density as a function of vacuum specific impulse with the circular orbit taken as a parameter. The power density $10$ is defined as the ratio of total jet power to the weight of the core, reflector, pressure shell, controls, and payload.

Figure 10 gives the results for a vehicle having a launch weight of 72,000 lb. The component weight breakdown for this vehicle is given in Table 3. Also included in Fig. 10 is a curve which demonstrates the gains to be made by tank staging. No attempt was made to do an optimum
Fig. 10 Calculations showing the effect of tank staging on payload, placed in a 100-mi orbit, as a function of reactor exit gas temperature (results for PAW trajectories; launch weight used in calculations was 72,000 lb)
TABLE 3

Circular Orbit Mission Results for a Vehicle Having a Launch Weight of 72,000 Lb

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Launch weight</td>
<td>72,000 lb</td>
</tr>
<tr>
<td>Flow rate</td>
<td>141.5 lb/sec</td>
</tr>
<tr>
<td>Reactor power</td>
<td>2214 Mw</td>
</tr>
<tr>
<td>Reactor exit gas temp</td>
<td>2170°C</td>
</tr>
<tr>
<td>Pump and turbine weight</td>
<td>700 lb</td>
</tr>
<tr>
<td>Nozzle weight</td>
<td>400 lb</td>
</tr>
<tr>
<td>Structure weight</td>
<td>1000 lb</td>
</tr>
<tr>
<td>Tank weight</td>
<td>4200 lb</td>
</tr>
<tr>
<td>Reactor weight</td>
<td>8000 lb</td>
</tr>
<tr>
<td>Fuel weight</td>
<td>53,100 lb</td>
</tr>
<tr>
<td>Holdover weight</td>
<td>600 lb</td>
</tr>
<tr>
<td>Tank diameter</td>
<td>14.5 ft</td>
</tr>
<tr>
<td>Ratio of over-all length to diameter</td>
<td>6</td>
</tr>
<tr>
<td>Payload weight</td>
<td>4000 lb</td>
</tr>
</tbody>
</table>
tank-staging calculation. The calculations were done by merely dropping a tank when the rocket reached an altitude of 200,000 ft. The weight of the tankage dropped was equivalent to the tank weight required to contain the fuel expended in reaching staging altitude. These calculations were performed to demonstrate that tank staging, if mechanically feasible, is quite advantageous.

2. Hydrogen-methane system. Experimentally it has been shown that graphite, at the temperatures of interest in nuclear rocket propulsion, is subject to severe corrosion by hydrogenous working fluids; and in order to achieve a useful long life of even the few minutes required by a rocket engine, some means of protecting the graphite surface is required.

Several approaches to this problem have been studied at LASL. One approach is to render the working fluid neutral towards hot graphite by the addition of some carbon-containing material such as methane, ethane, etc., to the hydrogenous working fluid. Another approach, which is showing promise, is to coat or line the surface of the graphite with niobium carbide. Results indicate that a working fluid of hydrogen and methane (3 mole %) at a temperature of 2100°C is possible without graphite corrosion. Without the methane the temperature of the
working fluid, for no graphite corrosion, is \( \approx 1500^\circ C \).\textsuperscript{12}

Preliminary experiments on coating (or lining) the graphite indicate the possibility of a hydrogen working fluid temperature of \( \approx 2500^\circ C \).\textsuperscript{13}

The technology of protecting the graphite-loaded fuel elements with niobium carbide has advanced to the point where the consideration of a working fluid of hydrogen-methane is no longer attractive. An example of the performance of a nuclear rocket using a working fluid of hydrogen and methane (3 mole \%) is given in Fig. 11 for an exit gas temperature of 1900\(^\circ\)C.

V. Elliptic Ascent

In these calculations the vehicle was flown from launch pad to height \( h_2 \) by means of a gravity turn and was then powered from \( h_2 \) into a 100-mi circular orbit in a manner that optimized payload. For purposes of discussion this 100-mi orbit will be referred to as orbit \( 1 \). The choice of the perigee orbit at 100 mi was due to practical considerations.\textsuperscript{14} The vehicle was flown from orbit \( 1 \) to various other circular orbits, all of which were at altitudes greater than 100 mi. The equations used for obtaining satellite payloads, for the elliptic ascent trajectory, will now be given. For this discussion we consider the flight of a vehicle between orbit \( 1 \) and
Fig. 11 Circular orbit radius as a function of payload for a vehicle containing a working fluid of hydrogen and methane (3 mole %) and operating at an exit gas temperature of 1900°C (results for PAW trajectories; launch weight used in calculations was 413,000 lb)
orbit 2, where the radius of orbit 2 is greater than orbit 1. The velocity increment required to transfer the rocket from circular orbit 1 to an elliptic orbit, with apogee at circular orbit 2, is

$$\Delta V_1 = V_p - V_{c1}$$  \hspace{1cm} (65)

where

- $V_p$ = the vehicle velocity at perigee in its elliptical orbit
- $V_{c1}$ = the vehicle velocity in circular orbit 1

The velocity increment required to transfer the vehicle from the apogee of the elliptic orbit into circular orbit 2 is

$$\Delta V_2 = V_{c2} - V_a$$  \hspace{1cm} (66)

where

- $V_a$ = the vehicle velocity at apogee in its elliptical orbit
- $V_{c2}$ = the vehicle velocity in circular orbit 2

The equation for the velocity of a vehicle in a circular orbit is

$$V_c = \left( \frac{g_0 R^2}{R} \right)^{1/2}$$  \hspace{1cm} (67)
where

\[ R = \text{the distance from the center of the earth to} \]
\[ \text{the vehicle (considered to be a mass point)} \]
\[ g_0 = 32.17 \text{ ft/sec}^2 \]
\[ R_e = \text{radius of the earth} = 2.0908 \times 10^7 \text{ ft} \]

The equation for the velocity of a vehicle in an elliptic orbit is\(^{15}\)

\[
V^2 = 2g_0 R_e^2 \left( \frac{1}{R} - \frac{1}{R_1 + R_2} \right)
\]

(68)

where

\[ R_1 = \text{the radius of orbit 1} \]
\[ R_2 = \text{the radius of orbit 2} \]

The total \( \Delta V \) required to go from orbit \( \text{1} \) to orbit \( \text{2} \), obtained by combining Eqs. 65 through 68, is

\[
\Delta V = V_{c1} \left( x^{-1/2} \left[ 1 - 2^{1/2} (1 - x) (1 + x)^{-1/2} \right]^{-1} \right)
\]

(69)

where

\[ x = \frac{\text{final orbit radius}}{\text{initial orbit radius}} \]

The burnout weight in orbit \( \text{2} \) can be obtained from a knowledge of the vehicle weight in orbit \( \text{1} \), the \( \Delta V \) required for the transfer, and the vehicle vacuum \( I_{sp} \). The equation relating these quantities is
\[ \Delta V = g_0 I_{sp} \ln \frac{R}{R} \quad (70) \]

where

\[ R = \frac{\text{weight in orbit 2}}{\text{weight in orbit 1}} \]

From a knowledge of the burnout weight in orbit 2 and the various component weights of the vehicle - tank weight, reactor weight, etc. - the payload in orbit 2 can be obtained. The results of this analysis are given in Table 4. The weight breakdown for flying flight 3 into a 300-mi orbit is given in Table 5.

In the Hohmann transfer part of the analysis it was assumed that the circular orbits were coplanar and that there were no gravitational losses.

VI. **Ballistic Ascent**

This trajectory was obtained from the previous work given in sections III and IV. The vehicle was flown straight up for a period of 16 sec, at which time it was kicked over to \( \beta_k \) and then flown to a specified height \( (h_{c.o.}) \). Upon reaching \( h_{c.o.} \) the thrust was turned off and the vehicle allowed to coast to summit, \( dh/dt = 0 \), at which time the thrust was cut back in and the vehicle powered into a 300-mi orbit. The optimization equations given in section IV were used to calculate the portion of the flight from summit to the 300-mi orbit.
TABLE 4
Optimum Trajectory Mission Results

<table>
<thead>
<tr>
<th>Flight</th>
<th>Temp, °C</th>
<th>I&lt;sub&gt;sp&lt;/sub&gt; vac</th>
<th>W&lt;sub&gt;R&lt;/sub&gt;, lb</th>
<th>P&lt;sub&gt;R,Mw&lt;/sub&gt;</th>
<th>Launch wt, lb</th>
<th>Payload in Orbit, lb</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100 mi</td>
<td>200 mi</td>
</tr>
<tr>
<td>1</td>
<td>1500</td>
<td>688</td>
<td>24,000</td>
<td>10,312</td>
<td>415,000</td>
<td>17,100</td>
</tr>
<tr>
<td>2</td>
<td>1900</td>
<td>761</td>
<td>22,000</td>
<td>11,752</td>
<td>413,000</td>
<td>34,200</td>
</tr>
<tr>
<td>3</td>
<td>2170</td>
<td>807</td>
<td>22,000</td>
<td>12,633</td>
<td>413,000</td>
<td>45,800</td>
</tr>
<tr>
<td>4</td>
<td>2500</td>
<td>860</td>
<td>21,000</td>
<td>13,669</td>
<td>413,000</td>
<td>56,100</td>
</tr>
<tr>
<td>5</td>
<td>3000</td>
<td>934</td>
<td>20,000</td>
<td>15,072</td>
<td>412,000</td>
<td>71,100</td>
</tr>
<tr>
<td>6</td>
<td>3500</td>
<td>1003</td>
<td>19,000</td>
<td>16,355</td>
<td>412,000</td>
<td>83,700</td>
</tr>
<tr>
<td>7</td>
<td>2170</td>
<td>807</td>
<td>8,000</td>
<td>2,214</td>
<td>72,000</td>
<td>4,000</td>
</tr>
<tr>
<td>8</td>
<td>2500</td>
<td>860</td>
<td>8,000</td>
<td>2,391</td>
<td>72,000</td>
<td>5,800</td>
</tr>
<tr>
<td>9*</td>
<td>2170</td>
<td>807</td>
<td>8,000</td>
<td>2,214</td>
<td>72,000</td>
<td>5,300</td>
</tr>
<tr>
<td>10*</td>
<td>2500</td>
<td>860</td>
<td>8,000</td>
<td>2,391</td>
<td>72,000</td>
<td>6,700</td>
</tr>
</tbody>
</table>

*Tank dropped at 200,000 ft.
<table>
<thead>
<tr>
<th>Description</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>H₂ load</td>
<td>308,000 lb</td>
</tr>
<tr>
<td>H₂ tank</td>
<td>24,000 lb</td>
</tr>
<tr>
<td>Structure</td>
<td>4,000 lb</td>
</tr>
<tr>
<td>Turbopump</td>
<td>9,000 lb</td>
</tr>
<tr>
<td>Nozzle</td>
<td>3,000 lb</td>
</tr>
<tr>
<td>Reactor and pressure shell</td>
<td>22,000 lb</td>
</tr>
<tr>
<td>Payload</td>
<td>43,000 lb</td>
</tr>
<tr>
<td><strong>Launch weight</strong></td>
<td><strong>413,000 lb</strong></td>
</tr>
<tr>
<td>Tank diameter</td>
<td>25 ft</td>
</tr>
<tr>
<td>Ratio of over-all length to</td>
<td>6</td>
</tr>
<tr>
<td>diameter</td>
<td></td>
</tr>
<tr>
<td>Flow rate</td>
<td>800 lb/sec</td>
</tr>
<tr>
<td>Reactor power</td>
<td>12,630 Mw</td>
</tr>
<tr>
<td>I&lt;sub&gt;sp&lt;/sub&gt; vac</td>
<td>807 sec</td>
</tr>
<tr>
<td>Dry weight</td>
<td>62,000 lb</td>
</tr>
</tbody>
</table>
To determine the most efficient ballistic trajectory from the standpoint of largest payload in the 300-mi orbit, curves of total vehicle weight in orbit as a function of $\beta_k$ were obtained with $h_{c.o.}$ as a parameter. From these curves a value for maximum total vehicle weight was determined for a given $h_{c.o.}$ and a curve of total vehicle weight (maximum) as a function of $h_{c.o.}$ was obtained. The results are given in Fig. 12. The vehicle launch weight for this problem was 413,000 lb and the reactor exit gas temperature used was 2170°C. These are the same values used for flight No. 3 of Table 4. A comparison of the payload for the ballistic ascent (41,000 lb) with that of the elliptic ascent (43,000 lb) shows that, for a 300-mi circular orbit, the elliptic ascent is slightly superior to the ballistic ascent.

VII. Conclusions

The trajectory calculations demonstrate quite clearly that, for orbits above 100 mi, the ballistic and elliptic ascents are far superior to the PAW ascent and that the elliptic ascent is superior to the ballistic ascent. In the elliptic and ballistic ascent calculations it was assumed that the thrust could be turned off and on during flight. This demonstrates the gains to be made by the use of a nuclear rocket motor which can be recycled.
Fig. 12 Total weight in pounds, placed in a 300-mi orbit as a function of thrustoff height in feet for a vehicle exit gas temperature of 2170°C, and a launch weight of 413,000 lb (results for a composite trajectory consisting of a gravity-turn part, ballistic trajectory to apogee, and then along an optimum path into a 300-mi orbit; results shown here are for an optimum value of $\beta_k$)
Another way of pointing out the effect the trajectory has on a given mission is in terms of power density. The power density requirement at 1900°C (see Fig. 9) for a 300-mi circular orbit mission drops from 0.452 Mw/lb to 0.208 Mw/lb in going from a PAW ascent to an elliptic ascent. The sensitivity of payload in orbit to temperature is also reduced when a vehicle is flown along an elliptic ascent trajectory into a low earth orbit (300 mi).

In section V it was shown that a single-stage vehicle having a launch weight of 413,000 lb and operating at a temperature of 2170°C could place 43,000 lb of payload into a 300-mi orbit. The three-stage Saturn-Titan-Centaur vehicle, having a launch weight of 1,075,000 lb, will place 25,000 lb into a 300-mi orbit. Replacing the Titan with a LOX-Hydrogen system will increase the payload to approximately 30,000 lb. The dry weight of the nuclear vehicle is 62,000 lb as compared to approximately 90,000 lb for the Saturn-Titan-Centaur vehicle.

A nuclear system having a launch weight of 72,000 lb, reactor power of 2214 Mw, and operating at an exit gas temperature of 2170°C (close to present day technology) will place 3500 lb of payload into a 300-mi orbit. Tank staging, see section V, will increase the payload to 4900 lb. The Atlas-Centaur plus a small solid-propellant
third stage will place approximately 7400 lb of payload into a 300-mi orbit. The dry weights of the nuclear and chemical systems are approximately the same.

Caution must be employed in drawing conclusions from the performances of the nuclear and chemical vehicles described above. The reason is that the performance of the chemical vehicle is based on multiple staging, whereas all the data presented for the nuclear systems are based on a single stage. An example of this point is that by going to a two-stage nuclear vehicle with a launch weight of 413,000 lb the payload can be increased from 43,000 lb to better than 70,000 lb. Circular orbit missions for two-stage nuclear systems will be given in a later report. The gains to be made in staging must also be weighed against a decrease in over-all reliability.

From the above performance data, but realizing the implications of comparing a three-stage chemical rocket with a single-stage nuclear rocket, one concludes that nuclear systems are superior to their chemical counterpart (same dry weight) for missions which require large payloads in orbit. The reverse is true for Atlas-Centaur type missions (small payloads). The reason for this is that the reactor weight does not decrease substantially with lowering power requirements; thus the power density
decreases, which in turn causes the percentage of dry weight which can go into payload to decrease. Future technology could very well lead to lower reactor weights, lower tank weights, and exit gas temperatures greater than 2170°C, thus leading to the increased performance of small-launch-weight nuclear systems. Present day technology points in this direction.

REFERENCES

2. W. Hohmann, Die Erreichbarkeit der Himmelskorper, Druck und Verlag R. Oldenbourg, Munich and Berlin, 1925.
12. H. Filip and A. A. Gonzalez, Internal document.
15. L. Page, Introduction to Theoretical Physics
    (2nd ed.), D. Van Nostrand Company, New York, 1947
    p. 95.
APPENDIX

FLOW DIAGRAM AND NUMERICAL METHOD

Input, box 1. Input numbers and identification are entered into the machine on six cards as indicated. On the first card any desired information (such as date, name of problem, etc.) may be punched in columns 2 through 72 inclusive. On the other five cards, numbers are punched, seven per card, in floating point form with ten columns per number. Each number has the format $+X.XXXXX+XX$. The decimal point is not punched but is assumed to be in the position shown. The last two digits along with the sign in the third last position are the power of 10 by which the first six digits (with point as shown) are multiplied. The nth number is punched in columns $10(n - 1) + 1$ through $10n$ inclusive, with the sign of the number in column $10(n - 1) + 1$ and the sign of the exponent of 10 in column $10(n - 1) + 8$.

Output. Output consists of a printed listing as shown in boxes 2, 4, and 13. Since the printer has only upper-case letters and no superscripts or subscripts, the table below is given to show the correspondence between the printing on the listing and the symbols used in the report and the flow diagram.
On Listing

Box 2:

BETAKD \( \beta_k \) (printed in degrees)
CAPK \( K \)
AO \( a_0 \)
BETAB \( B \)
TC \( T_c \)
FK \( k \)
CNL \( C_{NL} \)
AEAT \( A_e/A_t \)
PC \( P_c \)
FD \( f_d \)
WFO \( W_F^T \)
RHOT \( \rho_T \)
RHOH2 \( \rho_{H_2} \)
RHOCH4 \( \rho_{CH_4} \)
SMA \( a \)
FN1BAR \( \pi_1 \)
FN2BAR \( \pi_2 \)
ABAR \( A \)
CPH2 \( C_p \)
PD \( P_d \)
CN \( C_N \)
WL \( W_L \)
WR
FM1
FM2
VZ
TK
DELT1
ATCOEF
FM
THOFFH
THONH
H2F
HF
SETM4

Box 4:

WH2
WDOOTH2
WTH2
WSH2
WTP
WO
WCH4
WDOTCH
WTCH4
WSCH4
WN  \quad W_N
WDOT  \quad \dot{W}_F
F  \quad F
DT  \quad D_t
T  \quad t
V  \quad V
BETAD  \quad \beta \quad \text{(printed in degrees)}
H  \quad h
THETAD  \quad \theta \quad \text{(printed in degrees)}
TANACCEL  \quad \dot{V}
BETAPD  \quad \dot{\beta} \quad \text{(printed in degrees/sec)}
HP  \quad \dot{h}
THETAPD  \quad \dot{\theta} \quad \text{(printed in degrees/sec)}
L  \quad L
THRUST  \quad T
ISP  \quad I_{sp}

Box 13:
T  \quad t
L  \quad L
U  \quad U
H  \quad h
LAM1  \quad \lambda_1
LAM2  \quad \lambda_2
LAM3  \quad \lambda_3
W(T)  \quad W(t)
TH/W  \quad T/W(t)
V  \quad V
ACCEL (printed in degrees)
THETAD \( \theta \) (printed in degrees)
THETDD \( \dot{\theta} \) (printed in degrees/sec)
BETAD \( \beta \) (printed in degrees)
GAMMAD \( \gamma \) (printed in degrees)

Options. The various ways of running the problem are determined by the five input parameters \( W_T, h_{c.o.}, h_{c.on}, h_2, \) and SETM4. These variations center on when and how the thrust is turned off and on. The thrust is turned off either by having the missile run out of fuel (determined by \( W_T \)) or by specifying a cutoff height, \( h_{c.o.} \). It is turned on again either at the cut-on height, \( h_{c.on} \), or at the time when \( h = 0 \), depending on the value of SETM4. If \( SETM4 \neq 0 \), the thrust is turned on when \( h \geq h_{c.on} \), and if \( SETM4 = 0 \), the thrust is turned on when \( h \leq 0 \). The missile goes under control of the variational part when it reaches height \( h_2 \), provided the thrust is on.

The tabulation shows various possibilities. Here "normal" means that the quantity referred to is of a size which can be expected; for example, "normal" for \( h_{c.o.} \) would be a value of \( h \) at which it is reasonable to turn off the thrust, whereas "large" means larger than \( h \) can ever attain. "Immaterial" means that the corresponding quantity is never consulted.
<table>
<thead>
<tr>
<th>h\text{c.o.}</th>
<th>h\text{c.on}</th>
<th>h_2</th>
<th>SETM4</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>large</td>
<td>immaterial</td>
<td>large</td>
<td>immaterial</td>
<td>Missile runs out of fuel and finally hits ground.</td>
</tr>
<tr>
<td>large</td>
<td>immaterial</td>
<td>normal</td>
<td>immaterial</td>
<td>Missile goes up and into variational part when $h \geq h_2$ with thrust on all the time.</td>
</tr>
<tr>
<td>normal</td>
<td>large</td>
<td>large</td>
<td>\neq 0</td>
<td>Thrust is turned off when $h \geq h_{c.o.}'$, and missile finally hits ground.</td>
</tr>
<tr>
<td>normal</td>
<td>normal</td>
<td>normal</td>
<td>\neq 0</td>
<td>Thrust is turned off when $h \geq h_{c.o.}'$, is turned back on when $h \geq h_{c.on}'$, and missile goes into variational part when $h \geq h_2$.</td>
</tr>
<tr>
<td>normal</td>
<td>immaterial</td>
<td>normal</td>
<td>0</td>
<td>Thrust is turned off when $h \geq h_{c.o.}'$, is turned back on when $h = 0$, and missile goes into variational part when $h \geq h_2$.</td>
</tr>
<tr>
<td>normal</td>
<td>normal</td>
<td>large</td>
<td>\neq 0</td>
<td>Thrust is turned off when $h \geq h_{c.o.}'$, is turned back on when $h \geq h_{c.on}'$, and missile eventually runs out of fuel and finally hits ground.</td>
</tr>
<tr>
<td>normal</td>
<td>immaterial</td>
<td>large</td>
<td>0</td>
<td>Thrust is turned off when $h \geq h_{c.o.}'$, is turned back on when $h = 0$, and missile eventually runs out of fuel and finally hits ground.</td>
</tr>
</tbody>
</table>
Box 4

Print:

WH2 = WDOTH2 = WTH2 = WSH2 = WTP = WO =
WCH4 = WDOTCH4 = WTHCH4 = WSHCH4 = WN = WDOTH4 =
F = DT =
T V BETAD H THETAD TANACCEL BETAPD HP THETAPD L THRUST LSP

(Column headings)

Box 5

(Time dependent quantities)

\[ R = h + 20908000 \]
\[ P_a = 14.7 e^{-h/22800} \]
\[ R_e = 2.0905 \times 10^7 \]
\[ B = \begin{cases} 1 - \frac{P_a}{P} & \text{if } \frac{P_a}{P} \leq 3 \\ 2 & \text{if } \frac{P_a}{P} > 3 \end{cases} \]
\[ V_p = \frac{B}{A} \sqrt{P_a} \]
\[ t_0 = t - (\text{length of time thrust is off}) \]
\[ T_k = \frac{W}{T} \left( V_t + V \right) \left( 1 - \beta \right) \]
\[ \dot{W} = \frac{W}{W} - \dot{W} \frac{P}{P_B} \]
\[ P_a = (AT) e^{-h/22800} \]
\[ \alpha = \begin{cases} 1180 - 0.00414 h & \text{if } h < 25000 \\ 975 & \text{if } h \geq 25000 \end{cases} \]
\[ M = \frac{V}{C} \]
\[ C_D = \begin{cases} 2.9 + (M - 1) & \text{if } 0.3 < M < 1.1 \\ 3.9 + \frac{0.3}{M} & \text{if } 1.1 < M \end{cases} \]
\[ \frac{D_k}{W} = \frac{E C D \beta V^2}{W} \left( E \text{ found in Box 3} \right) \]
\[ \frac{e}{R^2} = \frac{E_0 B^2}{R} \]
\[ \text{Integrate from } t \text{ to } t + \Delta t: \]
\[ \dot{V} = \frac{T_k}{W} - \frac{D_k}{W} - \frac{g \cos \beta}{R} \]
\[ \dot{\beta} = \frac{V}{R} \sin \beta + \frac{V}{R} \sin \beta \]
\[ \dot{h} = V \cos \beta \]
\[ \dot{\theta} = \frac{V}{R} \sin \beta \]