Proof that Nonlinear Plane Waves cannot be destabilized by Scalar Diffusion

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We begin with the reaction–diffusion equation:
\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + D \nabla_\perp^2 u + F(u)
\]

where \( u \) is a vector of chemical concentrations and \( D \) is the diffusion matrix. Here \( \nabla_\perp^2 \) refers to directions orthogonal to \( x \). Assume that there exists a plane wave solution to Eq. (1). Denote this solution by \( U(x-ct) \). Define \( \xi = x - ct \). Then we have
\[
-cU_{\xi} = DU_{\xi \xi} + F(U)
\]

Now we ask under what conditions is the plane wave solution stable in multiple dimensions. To do this we work in the travelling coordinate system and write:
\[
u(\xi, y, t) = U(\xi) + \eta(\xi, y, t)
\]
where \( \eta \) is a small perturbation to the plane wave solution. We expand \( \eta \) in Fourier modes in \( y \). (The dimensionality of the system is irrelevant. For simplicity I work only in 2 space dimensions here, \( \xi \) and \( y \).) Thus we have:
\[
\eta = \sum_k a_k(\xi) \phi_k(y) \exp(\lambda_k t)
\]
where the \( \phi_k \) are eigenfunctions of \( \nabla_\perp^2 \). Inserting equations (3) and (4) into (1) we find the eigenvalue problem for \( \lambda_k \):
\[
\lambda_k a_k(\xi) = D \frac{\partial^2 a_k}{\partial \xi^2} - Dk^2 a_k(\xi) + J(\xi)a_k(\xi) + c \frac{\partial a_k}{\partial \xi}
\]

In the case that \( D = DI \) we have
\[
\lambda_k = \lambda_0 - Dk^2
\]

Equation (6) means that diffusion stabilizes small perturbations of the wave front in the case of scalar diffusion which is the desired result.