TITLE: RELATIVE SYMMETRIES OF DIFFERENTIAL EQUATIONS

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Let $\Delta : J^\infty \nu + J^\infty n$ be a differential operator, where $J^\infty \nu$ (resp. $J^\infty n$) is the infinite-jet bundle of the bundle $\nu : F \to M$ (resp. $n : E \to H$). Let $I^1_\nu$ be the Cartan submodule of the module $\Lambda^1(K_\nu)$ of 1-forms over the ring $K_\nu = C(J^\infty \nu)$. Among all derivations of $K_\nu$ into $K_\nu$ along $\Delta$, we classify those which map $I^1_\nu$ into $I^1_\nu$. They turn out to be quasi-evolution equations.

1. INTRODUCTION

Let $\pi : E \to M$, $\nu : F \to M$ be bundles (smooth, like everything else in the paper). Let $\pi_k : J^k \pi + M$, $\pi_k : J^k \nu + J^k n$ be the corresponding jet bundles, denote $J^\infty \pi = \lim \proj J^k \pi$, $K_\pi = C(J^\infty \pi) = \lim \ind C^0(J^k \pi)$. Let $\vec{\Delta} : J^\infty \nu + J^\infty n \to E$ be a bundle map (over $M$), which can be thought of as a differential operator $\vec{\Delta} : \Gamma(\nu) \to \Gamma(n)$, where $\Gamma(\nu)$ denotes the sheaf of sections of the bundle $\nu : F \to M$. Its annihilator in $\Lambda^1(J^\infty \nu)$ is the k-th Cartan submodule $I^1_k(\nu)$. The Cartan submodule $I^1(\nu)$ in $\Lambda^1(J^\infty \nu) = \Lambda^1(J^\infty \pi) = \lim \ind \Lambda^1(J^k \pi)$ is defined by the formula $I^1(\nu) = \lim \ind I^1_k(\nu)$. Let us denote by $\Delta$ the natural lift of $\vec{\Delta}$ into $J^\infty \nu$, $\Delta : J^\infty \nu + J^\infty n$. Then $\Delta^*(I^1_\pi) \subset I^1_\nu$ (lemma II 2.14 [3]).

We consider the following problem: find the set $\mathcal{D}^{\pi,\nu}(\Delta)$ of all derivations $Z : K_\pi \to K_\nu$ along the homomorphism $\Delta^*$, which map $I^1_\pi$ into $I^1_\nu$. There are at least three motivations for this problem:

A. In the case $\pi = \nu$, $\Delta = \id$, the set of all such $Z$'s is the set of evolution derivations $\mathcal{D}^{\pi,\nu}(\pi)$; in local coordinates, the equations of trajectories of these evolution derivations are evolution equations (Proposition 1 [2]; Theorem 1 5.6 [3]). (In the engineering literature, these derivations pass under the misleading name "Lie-Bäcklund transformations".)

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B. Such Z's arise in practice as the "generalized sine-Gordon equations" associated with classical simple complex Lie algebras ([4],[6]) and even with Kac-Moody Lie algebras ([11]).

C. Let \( U \subseteq J^n \) be a closed set considered as a differential equation: \( \gamma \in \Gamma(n) \) is a solution if \( (j_k(y))(M) \subseteq U \). Let \( \tilde{U} \subseteq J^{\infty} \) be the infinite prolongation of \( U \). Then the symmetries of \( \tilde{U} \) are those evolution derivations \( X \in L^e(n) \) which preserve the ideal \( \mathcal{F}(\tilde{U}) \) of functions from \( K \) vanishing on \( \tilde{U} \). Suppose, however, that \( \check{V} \subseteq J^{\infty} \) is another equation and \( \Delta(\check{V}) \subseteq \tilde{U} \). Then more general symmetries of \( \tilde{U} \) will be those Z's which map \( \mathcal{F}(\tilde{U}) \) into \( \mathcal{F}(\check{V}) \). That such relative symmetries are useful was demonstrated in a spectacular tour-de-force by Vinogradov and Krasil'shchik who used nonlocal symmetries to compute all (absolute) symmetries of the Korteweg-de Vries equation ([5]).

2. CLASSIFICATION

Denote by \( S(\pi) \) the \( K_1 \)-module of derivations of \( C^{\infty}(M) \) into \( K_1 \) along \( \pi \), where \( \pi : \xi^{\infty} + M \rightarrow M \) is the natural projection. Note that \( S(\pi) \) is generated over \( K_1 \) by the Lie algebra \( L(M) \) of vector fields on \( M \). If \( X \in L(\pi) \) then its lift \( \tilde{X} = \tilde{X} \in L(K_1) \) into the Lie algebra of derivations of \( K_1 \) is uniquely defined by the universal property \( j_\rho(y)^* \tilde{X} = j_\rho(y)^* Xj_\rho(y)^*, \forall y \in \Gamma(n) \), where \( \rho \) is such that \( X(C^{\infty}(M)) \subseteq C^{\infty}(J^{\infty}) \). The set of all such \( \tilde{X} \)'s is denoted by \( L(\pi) \) and is a Lie algebra and a \( K_1 \)-module (Theorem 1 3.6 [3]). The annihilator of \( L(\pi) \) in \( \Lambda^1(K_1) \) is nothing but the Cartan submodule \( \mathfrak{h} \). [This is the definition of the Cartan submodule; the fact that the corresponding distribution is spanned by the tangent planes of graphs of jets of sections of \( \pi \) is a corollary (Theorem 1 4.4 [3]).]

If \( X \in L(M) \) then the lifts \( \tilde{X} \) and \( \tilde{X} \) are \( \Delta \)-related: \( \tilde{X}_\Delta \Delta^* \tilde{X} = \Delta^* \tilde{X} \) (Lemma 1 2.13 [3]). Obviously, if \( X \in L(\pi) \), then again there exists a unique \( \tilde{X} \in L(\pi) \) such that \( \tilde{X}_\Delta \Delta^* \tilde{X} = \Delta^* \tilde{X} \); the resulting map \( L(\pi) \rightarrow L(\pi) \) is a Lie algebra homomorphism.

Lemma 2.1. Let \( \phi : K_1 + K_2 \rightarrow K_2 \) be a homomorphism of commutative rings \( K_1 \) and \( K_2 \), let \( X_1 \in L(K_1) \) and \( X_2 \in L(K_2) \) be two \( \phi \)-related derivations. Let \( L(\phi) \) be a \( K_2 \)-module of derivations of \( K_1 \) into \( K_2 \) along \( \phi \). Then for any \( Z \in L(\phi) \),

\[
(X_2 Z - ZX_1) \in L(\phi).
\]

Proof. Obvious.

Recall that if \( \omega \in \Lambda^1(K) \), \( X, Z \in L(K) \), then the Lie derivative of \( \omega \) with respect to \( Z \) is defined by the formula \( [Z(\omega)](X) = Z(\omega(X)) - \omega([Z,X]) \).

Lemma 2.2. In the notations of lemma 2.1, \( L(\phi) \) acts by derivations along \( \phi \) on \( \Lambda^1(K_1) \) with values in \( \Lambda^1(K_2) \). In particular, for \( \omega \in L^1(K_1) \)

\[
[Z(\omega)](X_2) = Z(\omega(X_1)) - \omega(ZX_1 - X_2 Z), \quad (2.3)
\]

where on the right hand side the pairing between \( \Lambda^1(K_1) \) and \( L(\phi) \) is understood naturally: \( (fdg)(Z) = \phi(f)Z(g) \), \( \forall f, g \in K_1 \).
Again, the proof is obvious.

Now we can handle the problem of classification of elements of $S^\text{qev}(\Delta)$. Let $Z \in S^\text{qev}(\Delta)$, that is, $Z(I^n) \subset I^n$. Take any $w \in \mathcal{I}^1_n = \text{Ann}(\mathcal{D}(\varpi))$. Then $Z(w) \in \mathcal{I}^1_n = \text{Ann}(\mathcal{D}(\varpi)) = \text{Ann}(\mathcal{D}(\varpi))$ iff, $\forall x \in \mathcal{H}$, $[Z(w)](\tilde{x}_i) = 0$. By formula (2.3), this is equivalent to $0 = Z(w(\tilde{x}_i)) - w(\tilde{x}_i \tilde{x}_i Z)$. But $w(\tilde{x}_i) = 0$ since we $1^n$. Thus $(\tilde{x}_i \tilde{x}_i Z) \in \text{Ann}(\mathcal{D}(\varpi))$.

Let $Z(\varpi) = Z(w) = 0$. By formula (2.3), this is equivalent to $0 = Z(u(\tilde{w})) - w(\tilde{w} \tilde{w} Z)$. But $Z(\varpi) = Z(w) = 0$ since we $1^n$. Thus $(\tilde{x}_i \tilde{x}_i Z) \in \text{Ann}(\mathcal{D}(\varpi))$.

\begin{equation}
(\tilde{x}_i \tilde{x}_i Z) \in \mathcal{D}(\varpi) \subset \mathcal{D}(\varpi).
\end{equation}

Theorem 2.5. Every $Z \in \mathcal{S}^\text{qev}(\Delta)$ is uniquely defined by its value $Z_{\varpi}$. Conversely, any derivation $Z_{\varpi} \in \mathcal{S}^\text{qev}(\Delta)$ is uniquely lifted in $\mathcal{S}(\phi)$ to become $Z_{\varpi} \in \mathcal{S}(\phi)$, such that $Z_{\varpi} = Z_{\varpi}$.

Proof. To study (2.4), first notice that, like in the absolute case ($\pi = \pi$, $\Delta = \pi$), one has a direct sum decomposition

\begin{equation}
\mathcal{D}(\pi) = \mathcal{D}(\varpi) \oplus \mathcal{D}(\varpi),
\end{equation}

where $\mathcal{D}(\pi) \oplus \mathcal{D}(\varpi)$, then $Z_1 = (Z_{\varpi}, Z_{\varpi}) \in \mathcal{D}(\varpi)$ and (2.4) for $Z = Z_1$ is obviously satisfied. Therefore we shall restrict ourselves to vertical $Z$'s $\mathcal{D}(\pi)$ only.

Let $(x_1, \ldots, x_m)$ be local coordinates in $M$, $(p_a) a = 1, \ldots, \text{dim } E = \text{dim } M$, $\sigma = Z_\alpha$ be standard local coordinates on $J^\pi$, and $(p^a) a = 1, \ldots, \text{dim } F - \text{dim } M$ be local coordinates on $J^\mu$. Let, locally, $Z = Z_\alpha^\alpha \Delta^\alpha \frac{\partial}{\partial q^\alpha} + A^\alpha \xi K^\alpha$. It is enough to check (2.4) for the basis vector fields $X = \frac{\partial}{\partial x_i} + q_i \frac{\partial}{\partial q^a}$ (using summation over repeated indices), we have

\begin{align*}
Z_\alpha^\alpha \Delta^\alpha \frac{\partial}{\partial q^\alpha} + A^\alpha \xi K^\alpha - (A^\alpha \frac{\partial}{\partial q^\alpha} + q_i \frac{\partial}{\partial q^a})& = \text{ since } \Delta^\alpha (\frac{\partial}{\partial x_i}) = (\frac{\partial}{\partial x_i})
\end{align*}

\begin{align*}
& = \{-((\frac{\partial}{\partial x_i}) (A^\alpha \frac{\partial}{\partial q^\alpha} + A^\alpha \xi) K^\alpha (\frac{\partial}{\partial x_i} \xi), (\frac{\partial}{\partial x_i} \xi))\}
\end{align*}
We have

\[-\frac{\partial}{\partial x_i} \left( A^a_0 + A^a_{i+1} \right) \Delta^* \frac{\partial}{\partial q^a_0} \]

This last expression must belong to $K_{\Delta^* S(M)}$. Since there are no components along $H$, it must vanish, and this happens iff $A^a_{i+1} = (D_i)_v (A^a_0)$, where $(D_i)_v$ stands for $(\partial/\partial x_i)_v$. Thus, $A^a_0 = (D^\sigma)_v (A^a_0)$, $(D^\sigma)_v : = (D^\sigma_1)_v \ldots (D^\sigma_m)_v$, and $A^a_0$s are arbitrary.

\section{Trajectories}

Ordinary differential equations are equations of trajectories of vector fields on manifolds. Analogously, evolution equations are equations of trajectories of vertical evolution derivations (Theorem 1 5.6 [3]). (The reason for considering only vertical fields is explained in §5 5.3 [3]; for nonvertical fields, equations become overdetermined.) Now let $Z \in \mathcal{Z}ev(\Delta)$, and consider $Z$ to be vertical. A trajectory of $Z$ is a one-parameter $(t)$ family of sections $\gamma = \gamma(t) : M \rightarrow F$ such that $[j(v)(\gamma)]^* Z = \frac{\partial}{\partial t} [j(n)(\Delta \gamma)]$. Let us find a coordinate version of the last equation. Let locally $Z = (D^\sigma)_v (A^a_0) \Delta^* \frac{\partial}{\partial q^a_0}$. Then

\[0 = \frac{\partial}{\partial t} [j(n)(\Delta \gamma)]^* Z = \frac{\partial}{\partial t} [j(n)(\Delta \gamma)]^* (D^\sigma)_v (A^a_0) \Delta^* \frac{\partial}{\partial q^a_0} \]

where $D^\sigma : = (\partial/\partial x_i^1) \ldots (\partial/\partial x_i^m)$. Since $[\partial/\partial t, D^\sigma] = 0$, the above equality is reduced to

\[\frac{\partial}{\partial t} [j(n)(\Delta \gamma)]^* (q^a_0) = [j(v)(\gamma)]^* (A^a_0) \]

Thus we obtain the coordinate form of quasievolution equations.

\textbf{Remark 3.2.} In contrast to the evolution equations, quasievolution ones need not be formally integrable. Obviously, integrability of a generic $Z$ depends only upon $\Delta$. I conjecture that this integrability depends only upon dimensions and codimensions of the finite number of prolongations of the map $\Delta : J^* V \rightarrow E$.

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BIBLIOGRAPHY


