Theory of a Radio-Frequency "Spin Filter" for a Metastable Hydrogen, Deuterium, or Tritium Atomic Beam
This report expresses the opinions of the author or authors and does not necessarily reflect the opinions or views of the Los Alamos Scientific Laboratory.
Theory of a Radio-Frequency “Spin Filter” for a Metastable Hydrogen, Deuterium, or Tritium Atomic Beam

by

Gerald G. Ohlsen
Joseph L. McKibben
CONTENTS

Abstract ............................................... 5
1. Introduction ......................................... 5
2. Energy Levels of the Hydrogen Atom ............... 6
3. Selection Rules ..................................... 8
4. Discussion of the General Spin State Selection
   Problem ............................................. 15
5. Quantum Mechanical Formulation of the Four-Level
   Problem ............................................. 16
6. Matrix Elements ..................................... 18
7. Analytic Solution of the Three-Level Problem .... 22
8. The Four-Level Problem ................................ 33
9. Adiabatic Variation of the Electric Fields ....... 40
   Acknowledgments .................................... 42
   References ......................................... 42
Appendix A: A Computer Program for Evaluating the
   Analytic Solution of the Three-Level
   Equations ........................................... 43
Appendix B: A Computer Program for Numerical Inte-
   gration of the Four-Level Equations .......... 49

TABLES

I. Parameters Characterizing the Hyperfine Structure
   of the n = 2 States of Hydrogen, Deuterium, and
   Tritium Atoms ..................................... 8
II. Hydrogen Atom 2S States ............................ 9
III. Hydrogen Atom 2P States ........................... 10
IV. Deuterium Atom 2S States .......................... 11
V. Deuterium Atom 2P States .......................... 12
VI. Tritium Atom 2S States ............................ 13
VII. Tritium Atom 2P States ........................... 14
VIII. n = 2 Hydrogen Atom Wave Functions in a
      Magnetic Field ................................... 19
IX. Factors Required for the Computation of the
    Hydrogen Atom Wave Functions in an Arbitrary
    Magnetic Field .................................... 20
X. n = 2 Electric Dipole Matrix Elements ........... 21
XI. Transmission of Unselected Substates ............. 27
THEORY OF A RADIO-FREQUENCY "SPIN FILTER" FOR A
METASTABLE HYDROGEN, DEUTERIUM, OR TRITIUM ATOMIC BEAM

by
Gerald G. Ohlsen and Joseph L. McKibben

ABSTRACT

Techniques for selection of metastable hydrogen, deuterium, or tritium atoms with a particular nuclear spin polarization are discussed. The emphasis is on the "three-level interaction" technique, which promises to be the most versatile and satisfactory of those available.

1. INTRODUCTION

In connection with the development of the Los Alamos Scientific Laboratory "metastable hydrogen" polarized ion source, calculations about possible nuclear spin selection techniques have been made. The discussion of these calculations, which have been partially reported, 1,2 comprises the main part of this report. However, for orientation purposes, the basic operating scheme for the LASL polarized ion source is reviewed. The discussion in this section is in terms of protons, although the scheme works as well for deuterons or for tritons.

A beam of protons is extracted from a duoplasmatron and then decelerated to a velocity (-3 x 10^7 cm/sec) corresponding to 500 eV. Additional electrons are injected to space-charge neutralize the beam, which is then passed through a curtain of cesium vapor. Collisions in the cesium convert a large number of protons into H_2S atoms, 3 together with ground-state atoms, and positive and negative ions. The beam then enters a longitudinal magnetic field of about 500 to 600 G. A transverse "clearing" electric field separates the charged particles from the neutral atoms (H_2S and H_1S). The neutral atoms then pass through a radio-frequency transition region where the H_2S atoms possessing any but the desired nuclear spin magnetic quantum number are quenched to the ground state while a large fraction of the atoms with the selected m_I remain in the H_2S state. This mixture of ~100% polarized H_2S and essentially unpolarized H_1S atoms is then passed through an argon exchange cell, where, at this particular velocity, the reaction H_2S + A → H + A occurs with a probability very much larger than does the corresponding ground-state reaction. 4

One interesting point is that the LASL nuclear spin selection method selects rather than rejects a particular nuclear spin state. Thus, a deuteron beam corresponding to a pure m_I = 1, 0, or -1 state may be obtained with a single radio-frequency selection device. The selection device, which employs perpendicular radio-frequency and static electric fields, behaves as a filter which allows only those metastable atoms with a specific nuclear spin quantum number (m_I) to pass through without being quenched to the ground state. This device is hereinafter referred to as a "spin filter."

The theoretical upper limit for transmission of the desired nuclear spin state through the spin filter is 1/2. Thus, for deuterons, at least 5/6 of the

The portion of the H_1S produced background current which arises from decay of H_2S atoms in the rf region is in fact partially polarized in the opposite sense.
incident metastable beam will be quenched, while for protons or tritons at least 3/4 of the incident metastable beam will be quenched. In addition, the incident atomic beam will have a large (perhaps 80 or 90%) ground-state component.

The degree of selection achieved in the argon exchange reaction depends on the ratio of the $^2S^+ + A \rightarrow H^- + A^+$ reaction cross section (denoted by $o^{2S}$) to the $^2S^+ + A \rightarrow H^- + A^+$ reaction cross section (denoted by $o^{1S}$). This ratio is not accurately known at present. The quantity which can be readily measured is the quenching ratio $Q$; that is, the ratio of the negative ion yield obtained through a cesium exchange reaction followed by an argon exchange reaction without and with the application of intervening fields sufficient to quench the entire metastable component of the beam. This ratio can be expressed in terms of the fraction of the atomic beam in the metastable state, $f$, as follows:

$$Q = (1 - f) + f(o^{2S}/o^{1S}).$$

The quenching ratio is related to the resulting nuclear spin state purity ($p$) by

$$p = 1 - 4/Qn$$

for protons and tritons, and by

$$p = 1 - 6/Qn$$

for deuterons, where $n$ is the efficiency of the spin filter. That is, $n = 1$ if the theoretical upper limit of $1/2$ for transmission is reached. The relationship between $p$ and the beam polarization parameters is given below.

### Spin 1/2 Particles

<table>
<thead>
<tr>
<th>$m_s$</th>
<th>$P$</th>
<th>$P_z$</th>
<th>$P_{zz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
</tr>
<tr>
<td>-1/2</td>
<td>$-p$</td>
<td>$-p$</td>
<td>$-p$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$m_s$</th>
<th>$P$</th>
<th>$P_z$</th>
<th>$P_{zz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p$</td>
<td>$p$</td>
<td>$p$</td>
</tr>
<tr>
<td>0</td>
<td>$0$</td>
<td>$0$</td>
<td>$-2p$</td>
</tr>
<tr>
<td>-1</td>
<td>$-p$</td>
<td>$-p$</td>
<td>$-p$</td>
</tr>
</tbody>
</table>

In the above $P = N(1/2) - N(-1/2)$, $P_z = N(1) - N(-1)$, and $P_{zz} = N(1) + N(-1) - 2N(0)$, where $N(m_s)$ is the fraction of the beam particles with quantum number $m_s$.

A measured value of $Q = 90$ was reported in Reference 4. This corresponds, for $n = 0.8$ (about the value expected), to a spin state purity corresponding to 91.7% for deuterons and 94.5% for protons and tritons. However, preliminary data obtained at LASL suggest that this value may be high by a factor of $\sim 2$. For intense beams, the achievable $Q$ seems to be further reduced to about 30. If these preliminary indications are correct, a spin state purity of about 83% for protons and tritons, and 75% for deuterons, would be expected.

### 2. ENERGY LEVELS OF THE HYDROGEN ATOM

We begin by reviewing briefly some facts about the $n = 2$ energy levels of the hydrogen atom. Figure 1 shows the $n = 2$ energy levels in a weak external magnetic field. At zero magnetic field, the energy difference between the $n = 2$ and $n = 1$ states is $13.6(\frac{1}{2} - \frac{1}{2})eV = 10.15$ eV. The $2S_{1/2} - 2P_{1/2}$ level spacing (the Lamb shift) corresponds to about 1059 MHz while the $2P_{3/2} - 2P_{1/2}$ level separation corresponds to about 10,968 MHz. In a weak magnetic field, the $2P_{3/2}$ states are split into four magnetic substates and the $2P_{1/2}$ and $2S_{1/2}$ levels each split into two magnetic substates. The $2P_{3/2}$ substates are usually referred to as $a$ (for $m_J = 3/2$), $b$ (for $m_J = 1/2$), $c$ (for $m_J = -1/2$), and $d$ (for $m_J = -3/2$).

![Fig. 1. The $n = 2$ levels of the hydrogen atom in a weak magnetic field; nuclear hyperfine structure is neglected.](image-url)
That is, there is a substate corresponding to each of the allowed nuclear magnetic quantum numbers $m_I = 1/2$ or $m_I = -1/2$. For deuterium atoms, where $m_I$ can have the values 1, 0, or -1, each substate is split into three nuclear magnetic substates. Note that the order of the $m_I$ substate energies is inverted when the electron spin $m_J$ value is negative.

The hyperfine energy level diagram for the $2S_{1/2}$ states is described by the Breit-Rabi formula:
\[
\Delta W = \frac{\Delta W}{2(2I+1)} + \frac{2m_J}{2I+1} X + \epsilon \Delta W \times m_F,
\]
where
\[
X = \frac{B}{B_1},
\]
\[
B_1 = \Delta W(1 + \epsilon)/(g_J \mu_0),
\]
\[
\epsilon = 1/(g_J \frac{1836.1 g_I}{g_I} - 1),
\]
\[
g_J = \text{Lande g factor},
\]
\[
\mu_0 = \text{Bohr magneton},
\]
\[
B = \text{magnetic field},
\]
\[
\Delta W = \text{zero field hyperfine splitting},
\]
\[
g_I = \mu_0 / I = \text{nuclear g factor}, \quad m_F = m_J + m_I.
\]

The last term arises from the interaction of the nuclear magnetic moment with the applied magnetic field and is, for ordinary magnetic fields, very small.

The Breit-Rabi formula is only approximate for the $2P_{1/2}$ levels since, for the field strengths of interest here, $J$ is only an approximately good quantum number. An exact calculation requires the diagonalization of the Hamiltonian including both fine structure and hyperfine structure terms; in the numerical results presented below we have only applied a first order correction to the Breit-Rabi formula by shifting the $e$ and $f$ lines downward by an amount calculated from the solution to the fine structure Zeeman splitting problem. Referred to the mean value of the multiplet, the correction is $\Delta W = -\frac{h}{9}(\frac{B}{2I+1} C)$. The values for the constants associated with the cases of interest are tabulated in Table I.
Table I
Parameters Characterizing the Hyperfine Structure of the n = 2 States
of Hydrogen, Deuterium, and Tritium Atoms

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>State</th>
<th>$\xi_J$</th>
<th>$\xi_I$</th>
<th>$\Delta$W(MHz)</th>
<th>$\epsilon$</th>
<th>$B_1^I$ (in G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton</td>
<td>$2S_{1/2}$</td>
<td>2.00229</td>
<td>5.585486</td>
<td>177.551</td>
<td>1.522 x 10^{-3}</td>
<td>63.448</td>
</tr>
<tr>
<td>Proton</td>
<td>$2P_{1/2}$</td>
<td>0.66589</td>
<td>5.585486</td>
<td>59.190</td>
<td>4.589 x 10^{-3}</td>
<td>63.796</td>
</tr>
<tr>
<td>Deuteron</td>
<td>$2S_{1/2}$</td>
<td>2.00229</td>
<td>0.857407</td>
<td>40.924</td>
<td>0.233 x 10^{-3}</td>
<td>14.605</td>
</tr>
<tr>
<td>Deuteron</td>
<td>$2P_{1/2}$</td>
<td>0.66589</td>
<td>0.857407</td>
<td>13.640</td>
<td>0.702 x 10^{-3}</td>
<td>14.644</td>
</tr>
<tr>
<td>Triton</td>
<td>$2S_{1/2}$</td>
<td>2.00229</td>
<td>5.957680</td>
<td>189.588</td>
<td>1.623 x 10^{-3}</td>
<td>67.755</td>
</tr>
<tr>
<td>Triton</td>
<td>$2P_{1/2}$</td>
<td>0.66589</td>
<td>5.957680</td>
<td>63.200</td>
<td>4.897 x 10^{-3}</td>
<td>68.138</td>
</tr>
</tbody>
</table>

In the numerical tabulations of the energy levels (Tables II-VII), all energies are expressed in equivalent frequency units (MHz) and are referred to the centroid of the $2S_{1/2}$ zero field multiplet. The latest published values of the Lamb shift have been used in this calculation ($1058.05$ MHz for H atoms and $1059.34$ MHz for D atoms). A value of $1058.05$ MHz has been used for T atoms.

The states are labeled by their strong field quantum numbers. Note that, for sufficiently high fields, the frequency separation for states whose $m_l$ values differ by 1 unit is $\Delta W/2$ for spin 1/2 particles and $\Delta W/3$ for spin 1 particles. Note also that the magnitude of the field $B_1$, which is customarily thought of as the field value which defines the weak and strong field regions, is much smaller here than in the case for ground-state atoms.

3. SELECTION RULES
The angular momenta involved in the complete description of a one-electron atomic state are the orbital angular momentum $\vec{I}$; the spin angular momentum $\vec{s}$; and the nuclear spin angular momentum $\vec{l}$. In a very strong magnetic field (i.e. in the Paschen-Back region, which is $-10^5$ 0 or greater for $n = 2$ hydrogen atoms) the quantities $\vec{I}$, $\vec{s}$, and $\vec{l}$ are completely decoupled. Thus, transitions may be induced which involve only one of the pairs of quantum numbers $l,m_l$; $s,m_s$; or $I,m_I$. In weaker fields where $\vec{l}$ and $\vec{s}$ couple to form $\vec{F}$ (usually known as the strong field region, which is in the range $-10^2$ to $-10^4$ G for $n = 2$ hydrogen atoms), transitions can be induced which involve either $J,m_J$ or $I,m_I$. Finally, for very weak fields, $\vec{J}$ and $\vec{F}$ couple to form $\vec{F}$, and transitions will involve changes in $F,m_F$. For a one-electron atom, $s$ is a constant; $I$ is always constant in atomic physics.

We are concerned here only with the lowest order transitions; i.e., with electric and magnetic dipole transitions.

a. Electric Dipole Selection Rules
The electric dipole operator is $e(\vec{E} \cdot \vec{r})$, where $e$ is the electronic charge, $\vec{E}$ is an applied (possibly oscillating) electric field, and $\vec{r}$ is the electron-nucleus radius vector. Since this operator is odd, it can have nonzero matrix elements only between states of opposite parity; i.e., $A\xi$ must be odd.

Further, if one expands the operator into spherical components, it can be shown (Section 6) that $A\xi = \pm 1$ is required.

For very strong fields, then, the selection rule is $A\xi = \pm 1$; $A\xi = 0, \pm 1$ and $A\xi = A\xi = 0$, since this operator does not affect the spin functions. For strong fields we have $A\xi = 0, \pm 1$; $A\xi = 0, \pm 1$ and $A\xi = 0$. For weak fields, we have $A\xi = 0, \pm 1$; $A\xi = 0, \pm 1$. If $A\xi = 0$, where $\xi$ represents whichever quantity is appropriate among $m_l$, $m_s$, $m_I$,
<table>
<thead>
<tr>
<th>GAUSS</th>
<th>X</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = -0.5 )</th>
<th>( \alpha = 0.5 )</th>
<th>( \alpha = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>44.39</td>
<td>44.39</td>
<td>44.39</td>
<td>133.16</td>
</tr>
<tr>
<td>20.0</td>
<td>0.315</td>
<td>72.46</td>
<td>48.69</td>
<td>16.72</td>
<td>-137.47</td>
</tr>
<tr>
<td>40.0</td>
<td>0.630</td>
<td>100.53</td>
<td>60.56</td>
<td>-11.75</td>
<td>-149.13</td>
</tr>
<tr>
<td>60.0</td>
<td>0.946</td>
<td>128.60</td>
<td>77.80</td>
<td>-29.82</td>
<td>-166.97</td>
</tr>
<tr>
<td>80.0</td>
<td>1.261</td>
<td>156.66</td>
<td>98.46</td>
<td>-67.69</td>
<td>-183.25</td>
</tr>
<tr>
<td>100.0</td>
<td>1.576</td>
<td>184.73</td>
<td>121.32</td>
<td>-95.96</td>
<td>-210.99</td>
</tr>
<tr>
<td>120.0</td>
<td>1.891</td>
<td>212.80</td>
<td>145.54</td>
<td>-124.93</td>
<td>-234.72</td>
</tr>
<tr>
<td>140.0</td>
<td>2.207</td>
<td>240.87</td>
<td>170.64</td>
<td>-152.10</td>
<td>-259.45</td>
</tr>
<tr>
<td>160.0</td>
<td>2.522</td>
<td>268.94</td>
<td>196.46</td>
<td>-180.17</td>
<td>-285.32</td>
</tr>
<tr>
<td>180.0</td>
<td>2.837</td>
<td>297.01</td>
<td>222.06</td>
<td>-209.24</td>
<td>-311.43</td>
</tr>
<tr>
<td>200.0</td>
<td>3.153</td>
<td>325.09</td>
<td>249.20</td>
<td>-236.46</td>
<td>-332.47</td>
</tr>
<tr>
<td>220.0</td>
<td>3.467</td>
<td>353.15</td>
<td>275.19</td>
<td>-264.17</td>
<td>-364.76</td>
</tr>
<tr>
<td>240.0</td>
<td>3.783</td>
<td>381.22</td>
<td>302.94</td>
<td>-292.94</td>
<td>-391.73</td>
</tr>
<tr>
<td>260.0</td>
<td>4.098</td>
<td>409.29</td>
<td>330.09</td>
<td>-320.91</td>
<td>-418.86</td>
</tr>
<tr>
<td>280.0</td>
<td>4.413</td>
<td>437.36</td>
<td>357.32</td>
<td>-349.65</td>
<td>-446.10</td>
</tr>
<tr>
<td>300.0</td>
<td>4.728</td>
<td>465.43</td>
<td>384.96</td>
<td>-377.65</td>
<td>-473.43</td>
</tr>
<tr>
<td>320.0</td>
<td>5.044</td>
<td>493.49</td>
<td>412.07</td>
<td>-404.72</td>
<td>-500.95</td>
</tr>
<tr>
<td>340.0</td>
<td>5.359</td>
<td>521.56</td>
<td>439.55</td>
<td>-432.79</td>
<td>-528.33</td>
</tr>
<tr>
<td>360.0</td>
<td>5.674</td>
<td>549.63</td>
<td>467.09</td>
<td>-460.86</td>
<td>-555.76</td>
</tr>
<tr>
<td>380.0</td>
<td>5.989</td>
<td>577.70</td>
<td>494.67</td>
<td>-488.93</td>
<td>-583.44</td>
</tr>
<tr>
<td>400.0</td>
<td>6.304</td>
<td>605.77</td>
<td>522.29</td>
<td>-517.00</td>
<td>-611.07</td>
</tr>
<tr>
<td>420.0</td>
<td>6.620</td>
<td>633.84</td>
<td>550.94</td>
<td>-545.07</td>
<td>-638.72</td>
</tr>
<tr>
<td>440.0</td>
<td>6.935</td>
<td>661.91</td>
<td>577.63</td>
<td>-573.13</td>
<td>-666.40</td>
</tr>
<tr>
<td>460.0</td>
<td>7.250</td>
<td>689.98</td>
<td>605.34</td>
<td>-601.20</td>
<td>-694.07</td>
</tr>
<tr>
<td>480.0</td>
<td>7.565</td>
<td>718.05</td>
<td>633.07</td>
<td>-629.27</td>
<td>-721.45</td>
</tr>
<tr>
<td>500.0</td>
<td>7.881</td>
<td>746.12</td>
<td>660.82</td>
<td>-657.34</td>
<td>-749.06</td>
</tr>
<tr>
<td>520.0</td>
<td>8.196</td>
<td>774.19</td>
<td>688.59</td>
<td>-685.41</td>
<td>-777.47</td>
</tr>
<tr>
<td>540.0</td>
<td>8.511</td>
<td>802.26</td>
<td>716.38</td>
<td>-713.49</td>
<td>-805.15</td>
</tr>
<tr>
<td>560.0</td>
<td>8.826</td>
<td>830.33</td>
<td>744.18</td>
<td>-741.55</td>
<td>-832.88</td>
</tr>
<tr>
<td>580.0</td>
<td>9.141</td>
<td>858.39</td>
<td>771.99</td>
<td>-769.62</td>
<td>-860.57</td>
</tr>
<tr>
<td>600.0</td>
<td>9.457</td>
<td>886.46</td>
<td>799.91</td>
<td>-797.69</td>
<td>-888.29</td>
</tr>
<tr>
<td>620.0</td>
<td>9.772</td>
<td>914.53</td>
<td>827.65</td>
<td>-825.76</td>
<td>-916.42</td>
</tr>
<tr>
<td>640.0</td>
<td>10.087</td>
<td>942.60</td>
<td>855.43</td>
<td>-853.83</td>
<td>-944.27</td>
</tr>
<tr>
<td>660.0</td>
<td>10.402</td>
<td>970.67</td>
<td>883.33</td>
<td>-881.90</td>
<td>-972.12</td>
</tr>
<tr>
<td>680.0</td>
<td>10.717</td>
<td>998.74</td>
<td>911.20</td>
<td>-909.97</td>
<td>-999.05</td>
</tr>
<tr>
<td>700.0</td>
<td>11.033</td>
<td>1026.81</td>
<td>939.07</td>
<td>-937.03</td>
<td>-1027.46</td>
</tr>
<tr>
<td>720.0</td>
<td>11.348</td>
<td>1054.88</td>
<td>966.94</td>
<td>-965.10</td>
<td>-1055.72</td>
</tr>
<tr>
<td>740.0</td>
<td>11.663</td>
<td>1082.95</td>
<td>994.82</td>
<td>-994.17</td>
<td>-1083.59</td>
</tr>
<tr>
<td>760.0</td>
<td>11.978</td>
<td>1111.02</td>
<td>1022.70</td>
<td>-1022.24</td>
<td>-1111.44</td>
</tr>
<tr>
<td>780.0</td>
<td>12.294</td>
<td>1139.09</td>
<td>1050.59</td>
<td>-1050.62</td>
<td>-1139.17</td>
</tr>
<tr>
<td>800.0</td>
<td>12.609</td>
<td>1167.16</td>
<td>1078.49</td>
<td>-1078.88</td>
<td>-1167.36</td>
</tr>
<tr>
<td>820.0</td>
<td>12.924</td>
<td>1195.22</td>
<td>1106.38</td>
<td>-1106.38</td>
<td>-1195.35</td>
</tr>
<tr>
<td>840.0</td>
<td>13.239</td>
<td>1223.29</td>
<td>1134.29</td>
<td>-1134.29</td>
<td>-1223.07</td>
</tr>
<tr>
<td>860.0</td>
<td>13.555</td>
<td>1251.36</td>
<td>1162.20</td>
<td>-1162.59</td>
<td>-1250.97</td>
</tr>
<tr>
<td>880.0</td>
<td>13.870</td>
<td>1279.43</td>
<td>1190.11</td>
<td>-1190.46</td>
<td>-1278.84</td>
</tr>
<tr>
<td>900.0</td>
<td>14.185</td>
<td>1307.50</td>
<td>1218.02</td>
<td>-1218.33</td>
<td>-1306.70</td>
</tr>
<tr>
<td>920.0</td>
<td>14.500</td>
<td>1335.57</td>
<td>1245.94</td>
<td>-1245.94</td>
<td>-1334.71</td>
</tr>
<tr>
<td>940.0</td>
<td>14.815</td>
<td>1363.64</td>
<td>1273.86</td>
<td>-1273.86</td>
<td>-1362.73</td>
</tr>
<tr>
<td>960.0</td>
<td>15.131</td>
<td>1391.71</td>
<td>1301.78</td>
<td>-1301.78</td>
<td>-1390.77</td>
</tr>
<tr>
<td>980.0</td>
<td>15.446</td>
<td>1419.78</td>
<td>1329.70</td>
<td>-1329.70</td>
<td>-1418.84</td>
</tr>
<tr>
<td>1000.0</td>
<td>15.761</td>
<td>1447.85</td>
<td>1357.63</td>
<td>-1357.63</td>
<td>-1446.40</td>
</tr>
</tbody>
</table>
### TABLE III

**HYDROGEN ATOM 2P STATES**

<table>
<thead>
<tr>
<th>GAUSS</th>
<th>X</th>
<th>MI=n=5</th>
<th>MI=n=5</th>
<th>MI=0=5</th>
<th>MI=n=5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
<td>-1043.25</td>
<td>-1043.25</td>
<td>-1043.25</td>
<td>-1102.44</td>
</tr>
<tr>
<td>20.0</td>
<td>.314</td>
<td>-1033.93</td>
<td>-1041.86</td>
<td>-1052.63</td>
<td>-1103.94</td>
</tr>
<tr>
<td>40.0</td>
<td>.627</td>
<td>-1024.64</td>
<td>-1037.98</td>
<td>-1061.99</td>
<td>-1107.95</td>
</tr>
<tr>
<td>60.0</td>
<td>.941</td>
<td>-1015.38</td>
<td>-1032.35</td>
<td>-1071.41</td>
<td>-1113.76</td>
</tr>
<tr>
<td>80.0</td>
<td>1.254</td>
<td>-1006.16</td>
<td>-1025.60</td>
<td>-1080.86</td>
<td>-1120.74</td>
</tr>
<tr>
<td>100.0</td>
<td>1.568</td>
<td>-996.96</td>
<td>-1018.18</td>
<td>-1090.34</td>
<td>-1128.47</td>
</tr>
<tr>
<td>120.0</td>
<td>1.881</td>
<td>-987.80</td>
<td>-1010.29</td>
<td>-1099.34</td>
<td>-1136.49</td>
</tr>
<tr>
<td>140.0</td>
<td>2.195</td>
<td>-978.67</td>
<td>-1002.15</td>
<td>-1109.40</td>
<td>-1145.25</td>
</tr>
<tr>
<td>160.0</td>
<td>2.508</td>
<td>-969.57</td>
<td>-993.83</td>
<td>-1118.98</td>
<td>-1154.05</td>
</tr>
<tr>
<td>180.0</td>
<td>2.822</td>
<td>-960.50</td>
<td>-985.39</td>
<td>-1128.49</td>
<td>-1163.03</td>
</tr>
<tr>
<td>200.0</td>
<td>3.135</td>
<td>-951.57</td>
<td>-976.87</td>
<td>-1138.93</td>
<td>-1172.56</td>
</tr>
<tr>
<td>220.0</td>
<td>3.449</td>
<td>-942.64</td>
<td>-968.30</td>
<td>-1148.39</td>
<td>-1182.18</td>
</tr>
<tr>
<td>240.0</td>
<td>3.762</td>
<td>-933.74</td>
<td>-959.70</td>
<td>-1158.81</td>
<td>-1191.81</td>
</tr>
<tr>
<td>260.0</td>
<td>4.076</td>
<td>-924.85</td>
<td>-951.08</td>
<td>-1167.35</td>
<td>-1201.07</td>
</tr>
<tr>
<td>280.0</td>
<td>4.389</td>
<td>-915.98</td>
<td>-941.46</td>
<td>-1177.12</td>
<td>-1209.60</td>
</tr>
<tr>
<td>300.0</td>
<td>4.703</td>
<td>-907.11</td>
<td>-931.82</td>
<td>-1186.92</td>
<td>-1219.15</td>
</tr>
<tr>
<td>320.0</td>
<td>5.016</td>
<td>-898.26</td>
<td>-922.19</td>
<td>-1196.75</td>
<td>-1229.57</td>
</tr>
<tr>
<td>340.0</td>
<td>5.329</td>
<td>-890.42</td>
<td>-912.59</td>
<td>-1206.52</td>
<td>-1239.94</td>
</tr>
<tr>
<td>360.0</td>
<td>5.643</td>
<td>-882.59</td>
<td>-902.99</td>
<td>-1216.31</td>
<td>-1250.12</td>
</tr>
<tr>
<td>380.0</td>
<td>5.957</td>
<td>-874.77</td>
<td>-893.39</td>
<td>-1226.14</td>
<td>-1259.27</td>
</tr>
<tr>
<td>400.0</td>
<td>6.270</td>
<td>-867.00</td>
<td>-883.80</td>
<td>-1236.00</td>
<td>-1267.77</td>
</tr>
<tr>
<td>420.0</td>
<td>6.584</td>
<td>-859.24</td>
<td>-874.21</td>
<td>-1245.99</td>
<td>-1277.51</td>
</tr>
<tr>
<td>440.0</td>
<td>6.897</td>
<td>-851.50</td>
<td>-864.63</td>
<td>-1255.92</td>
<td>-1287.31</td>
</tr>
<tr>
<td>460.0</td>
<td>7.211</td>
<td>-843.78</td>
<td>-855.06</td>
<td>-1266.86</td>
<td>-1297.12</td>
</tr>
<tr>
<td>480.0</td>
<td>7.524</td>
<td>-836.08</td>
<td>-845.53</td>
<td>-1277.81</td>
<td>-1307.09</td>
</tr>
<tr>
<td>500.0</td>
<td>7.838</td>
<td>-828.40</td>
<td>-835.99</td>
<td>-1288.79</td>
<td>-1317.89</td>
</tr>
<tr>
<td>520.0</td>
<td>8.151</td>
<td>-820.74</td>
<td>-825.41</td>
<td>-1299.72</td>
<td>-1328.84</td>
</tr>
<tr>
<td>540.0</td>
<td>8.465</td>
<td>-813.10</td>
<td>-814.83</td>
<td>-1310.67</td>
<td>-1339.94</td>
</tr>
<tr>
<td>560.0</td>
<td>8.778</td>
<td>-805.47</td>
<td>-804.27</td>
<td>-1321.65</td>
<td>-1351.11</td>
</tr>
<tr>
<td>580.0</td>
<td>9.092</td>
<td>-797.85</td>
<td>-793.71</td>
<td>-1332.63</td>
<td>-1362.36</td>
</tr>
<tr>
<td>600.0</td>
<td>9.405</td>
<td>-790.25</td>
<td>-783.16</td>
<td>-1343.63</td>
<td>-1373.71</td>
</tr>
<tr>
<td>620.0</td>
<td>9.719</td>
<td>-782.66</td>
<td>-772.62</td>
<td>-1354.63</td>
<td>-1385.16</td>
</tr>
<tr>
<td>640.0</td>
<td>10.032</td>
<td>-775.08</td>
<td>-763.09</td>
<td>-1365.65</td>
<td>-1396.69</td>
</tr>
<tr>
<td>660.0</td>
<td>10.346</td>
<td>-767.51</td>
<td>-753.57</td>
<td>-1376.69</td>
<td>-1408.23</td>
</tr>
<tr>
<td>680.0</td>
<td>10.659</td>
<td>-760.05</td>
<td>-744.06</td>
<td>-1387.75</td>
<td>-1419.85</td>
</tr>
<tr>
<td>700.0</td>
<td>10.973</td>
<td>-752.60</td>
<td>-734.53</td>
<td>-1398.83</td>
<td>-1431.52</td>
</tr>
<tr>
<td>720.0</td>
<td>11.286</td>
<td>-745.17</td>
<td>-725.01</td>
<td>-1409.94</td>
<td>-1443.25</td>
</tr>
<tr>
<td>740.0</td>
<td>11.600</td>
<td>-737.77</td>
<td>-715.51</td>
<td>-1421.09</td>
<td>-1455.11</td>
</tr>
<tr>
<td>760.0</td>
<td>11.913</td>
<td>-720.38</td>
<td>-706.02</td>
<td>-1432.27</td>
<td>-1466.97</td>
</tr>
<tr>
<td>780.0</td>
<td>12.227</td>
<td>-713.00</td>
<td>-696.55</td>
<td>-1443.54</td>
<td>-1478.95</td>
</tr>
<tr>
<td>800.0</td>
<td>12.540</td>
<td>-705.64</td>
<td>-687.12</td>
<td>-1454.85</td>
<td>-1490.94</td>
</tr>
<tr>
<td>820.0</td>
<td>12.854</td>
<td>-698.29</td>
<td>-677.71</td>
<td>-1466.18</td>
<td>-1503.01</td>
</tr>
<tr>
<td>840.0</td>
<td>13.167</td>
<td>-690.96</td>
<td>-668.31</td>
<td>-1477.52</td>
<td>-1515.14</td>
</tr>
<tr>
<td>860.0</td>
<td>13.481</td>
<td>-683.74</td>
<td>-658.92</td>
<td>-1488.89</td>
<td>-1527.32</td>
</tr>
<tr>
<td>880.0</td>
<td>13.794</td>
<td>-676.54</td>
<td>-649.54</td>
<td>-1490.27</td>
<td>-1539.53</td>
</tr>
<tr>
<td>900.0</td>
<td>14.108</td>
<td>-669.35</td>
<td>-640.18</td>
<td>-1501.69</td>
<td>-1551.71</td>
</tr>
<tr>
<td>920.0</td>
<td>14.421</td>
<td>-662.17</td>
<td>-630.83</td>
<td>-1513.14</td>
<td>-1563.94</td>
</tr>
<tr>
<td>940.0</td>
<td>14.735</td>
<td>-654.99</td>
<td>-621.49</td>
<td>-1524.61</td>
<td>-1576.17</td>
</tr>
<tr>
<td>960.0</td>
<td>15.048</td>
<td>-647.83</td>
<td>-612.16</td>
<td>-1536.12</td>
<td>-1588.42</td>
</tr>
<tr>
<td>980.0</td>
<td>15.362</td>
<td>-640.68</td>
<td>-602.84</td>
<td>-1547.66</td>
<td>-1600.71</td>
</tr>
<tr>
<td>1000.0</td>
<td>15.675</td>
<td>-633.55</td>
<td>-593.53</td>
<td>-1559.22</td>
<td>-1613.02</td>
</tr>
</tbody>
</table>

*Note: The table continues with similar entries for different values of GAUSS.*
## Table IV

<table>
<thead>
<tr>
<th>GAUSS</th>
<th>X</th>
<th>M³⁺</th>
<th>M⁺⁺</th>
<th>M⁺⁺⁺</th>
<th>M⁻⁻</th>
<th>M⁻⁻⁻</th>
<th>M⁻⁻⁻⁻</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>13.64</td>
<td>13.64</td>
<td>13.64</td>
<td>-27.28</td>
<td>-27.28</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.150</td>
<td>41.68</td>
<td>-35.49</td>
<td>46.64</td>
<td>-59.70</td>
<td>72.56</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>0.300</td>
<td>69.72</td>
<td>21.44</td>
<td>-19.40</td>
<td>-35.49</td>
<td>46.64</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>0.450</td>
<td>97.76</td>
<td>46.30</td>
<td>-42.44</td>
<td>-59.70</td>
<td>72.56</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.600</td>
<td>125.80</td>
<td>-98.62</td>
<td>-113.86</td>
<td>-127.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>0.750</td>
<td>153.84</td>
<td>127.82</td>
<td>-126.56</td>
<td>-141.52</td>
<td>-154.97</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>0.900</td>
<td>181.88</td>
<td>155.59</td>
<td>-154.60</td>
<td>-169.31</td>
<td>-182.78</td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>1.050</td>
<td>209.92</td>
<td>193.43</td>
<td>-182.64</td>
<td>-197.17</td>
<td>-210.65</td>
<td></td>
</tr>
<tr>
<td>8.0</td>
<td>1.200</td>
<td>237.96</td>
<td>231.12</td>
<td>-210.64</td>
<td>-225.14</td>
<td>-238.60</td>
<td></td>
</tr>
<tr>
<td>9.0</td>
<td>1.350</td>
<td>266.00</td>
<td>269.74</td>
<td>-238.72</td>
<td>-254.00</td>
<td>-266.48</td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>1.500</td>
<td>294.04</td>
<td>297.17</td>
<td>-266.76</td>
<td>-282.05</td>
<td>-294.42</td>
<td></td>
</tr>
<tr>
<td>11.0</td>
<td>1.650</td>
<td>322.08</td>
<td>-295.12</td>
<td>-311.40</td>
<td>-327.78</td>
<td>-344.15</td>
<td></td>
</tr>
<tr>
<td>12.0</td>
<td>1.800</td>
<td>350.12</td>
<td>323.09</td>
<td>-329.76</td>
<td>-346.13</td>
<td>-362.50</td>
<td></td>
</tr>
<tr>
<td>13.0</td>
<td>1.950</td>
<td>378.16</td>
<td>351.16</td>
<td>-346.57</td>
<td>-363.87</td>
<td>-380.52</td>
<td></td>
</tr>
<tr>
<td>14.0</td>
<td>2.100</td>
<td>406.20</td>
<td>379.03</td>
<td>-363.47</td>
<td>-381.57</td>
<td>-401.15</td>
<td></td>
</tr>
<tr>
<td>15.0</td>
<td>2.250</td>
<td>434.24</td>
<td>407.01</td>
<td>-380.37</td>
<td>-401.57</td>
<td>-421.15</td>
<td></td>
</tr>
<tr>
<td>16.0</td>
<td>2.400</td>
<td>462.28</td>
<td>434.24</td>
<td>-397.27</td>
<td>-419.57</td>
<td>-448.15</td>
<td></td>
</tr>
<tr>
<td>17.0</td>
<td>2.550</td>
<td>490.32</td>
<td>462.24</td>
<td>-414.17</td>
<td>-437.57</td>
<td>-467.15</td>
<td></td>
</tr>
<tr>
<td>18.0</td>
<td>2.700</td>
<td>518.36</td>
<td>490.98</td>
<td>-431.07</td>
<td>-456.05</td>
<td>-486.15</td>
<td></td>
</tr>
<tr>
<td>19.0</td>
<td>2.850</td>
<td>546.40</td>
<td>518.97</td>
<td>-447.97</td>
<td>-474.55</td>
<td>-506.15</td>
<td></td>
</tr>
<tr>
<td>20.0</td>
<td>3.000</td>
<td>574.44</td>
<td>546.97</td>
<td>-464.87</td>
<td>-494.05</td>
<td>-526.15</td>
<td></td>
</tr>
<tr>
<td>21.0</td>
<td>3.150</td>
<td>602.47</td>
<td>574.96</td>
<td>-481.77</td>
<td>-513.55</td>
<td>-556.15</td>
<td></td>
</tr>
<tr>
<td>22.0</td>
<td>3.300</td>
<td>630.51</td>
<td>602.96</td>
<td>-508.67</td>
<td>-533.05</td>
<td>-586.15</td>
<td></td>
</tr>
<tr>
<td>23.0</td>
<td>3.450</td>
<td>658.55</td>
<td>630.96</td>
<td>-525.57</td>
<td>-552.55</td>
<td>-598.15</td>
<td></td>
</tr>
<tr>
<td>24.0</td>
<td>3.600</td>
<td>686.59</td>
<td>658.96</td>
<td>-542.47</td>
<td>-572.05</td>
<td>-628.15</td>
<td></td>
</tr>
<tr>
<td>25.0</td>
<td>3.750</td>
<td>714.63</td>
<td>686.97</td>
<td>-559.37</td>
<td>-591.55</td>
<td>-658.15</td>
<td></td>
</tr>
<tr>
<td>26.0</td>
<td>3.900</td>
<td>742.67</td>
<td>714.97</td>
<td>-576.27</td>
<td>-611.05</td>
<td>-688.15</td>
<td></td>
</tr>
<tr>
<td>27.0</td>
<td>4.050</td>
<td>770.71</td>
<td>742.97</td>
<td>-593.17</td>
<td>-630.55</td>
<td>-718.15</td>
<td></td>
</tr>
<tr>
<td>28.0</td>
<td>4.200</td>
<td>798.75</td>
<td>770.98</td>
<td>-610.07</td>
<td>-650.05</td>
<td>-748.15</td>
<td></td>
</tr>
<tr>
<td>29.0</td>
<td>4.350</td>
<td>826.79</td>
<td>800.98</td>
<td>-626.97</td>
<td>-669.55</td>
<td>-778.15</td>
<td></td>
</tr>
<tr>
<td>30.0</td>
<td>4.500</td>
<td>854.83</td>
<td>830.98</td>
<td>-643.87</td>
<td>-689.05</td>
<td>-808.15</td>
<td></td>
</tr>
<tr>
<td>31.0</td>
<td>4.650</td>
<td>882.87</td>
<td>860.98</td>
<td>-660.77</td>
<td>-708.55</td>
<td>-838.15</td>
<td></td>
</tr>
<tr>
<td>32.0</td>
<td>4.800</td>
<td>910.91</td>
<td>890.98</td>
<td>-677.67</td>
<td>-728.05</td>
<td>-868.15</td>
<td></td>
</tr>
<tr>
<td>33.0</td>
<td>4.950</td>
<td>938.95</td>
<td>920.98</td>
<td>-694.57</td>
<td>-747.55</td>
<td>-898.15</td>
<td></td>
</tr>
<tr>
<td>34.0</td>
<td>5.100</td>
<td>966.99</td>
<td>950.98</td>
<td>-711.47</td>
<td>-767.05</td>
<td>-928.15</td>
<td></td>
</tr>
<tr>
<td>35.0</td>
<td>5.250</td>
<td>995.03</td>
<td>980.98</td>
<td>-728.37</td>
<td>-786.55</td>
<td>-958.15</td>
<td></td>
</tr>
<tr>
<td>36.0</td>
<td>5.400</td>
<td>1023.07</td>
<td>1010.98</td>
<td>-745.27</td>
<td>-806.05</td>
<td>-988.15</td>
<td></td>
</tr>
<tr>
<td>37.0</td>
<td>5.550</td>
<td>1051.11</td>
<td>1039.98</td>
<td>-762.17</td>
<td>-825.55</td>
<td>-1018.15</td>
<td></td>
</tr>
<tr>
<td>38.0</td>
<td>5.700</td>
<td>1079.15</td>
<td>1069.98</td>
<td>-779.07</td>
<td>-845.05</td>
<td>-1048.15</td>
<td></td>
</tr>
<tr>
<td>39.0</td>
<td>5.850</td>
<td>1107.19</td>
<td>1099.98</td>
<td>-795.97</td>
<td>-864.55</td>
<td>-1078.15</td>
<td></td>
</tr>
<tr>
<td>40.0</td>
<td>6.000</td>
<td>1135.23</td>
<td>1129.98</td>
<td>-812.87</td>
<td>-884.05</td>
<td>-1108.15</td>
<td></td>
</tr>
<tr>
<td>41.0</td>
<td>6.150</td>
<td>1163.27</td>
<td>1159.98</td>
<td>-829.77</td>
<td>-903.55</td>
<td>-1138.15</td>
<td></td>
</tr>
<tr>
<td>42.0</td>
<td>6.300</td>
<td>1191.31</td>
<td>1189.98</td>
<td>-846.67</td>
<td>-923.05</td>
<td>-1168.15</td>
<td></td>
</tr>
<tr>
<td>43.0</td>
<td>6.450</td>
<td>1219.35</td>
<td>1219.98</td>
<td>-863.57</td>
<td>-942.55</td>
<td>-1198.15</td>
<td></td>
</tr>
<tr>
<td>44.0</td>
<td>6.600</td>
<td>1247.39</td>
<td>1249.98</td>
<td>-880.47</td>
<td>-962.05</td>
<td>-1228.15</td>
<td></td>
</tr>
<tr>
<td>45.0</td>
<td>6.750</td>
<td>1275.43</td>
<td>1279.98</td>
<td>-897.37</td>
<td>-981.55</td>
<td>-1258.15</td>
<td></td>
</tr>
<tr>
<td>46.0</td>
<td>6.900</td>
<td>1303.47</td>
<td>1309.98</td>
<td>-914.27</td>
<td>-1001.05</td>
<td>-1288.15</td>
<td></td>
</tr>
<tr>
<td>47.0</td>
<td>7.050</td>
<td>1331.51</td>
<td>1339.98</td>
<td>-931.17</td>
<td>-1020.55</td>
<td>-1318.15</td>
<td></td>
</tr>
<tr>
<td>48.0</td>
<td>7.200</td>
<td>1359.55</td>
<td>1369.98</td>
<td>-948.07</td>
<td>-1040.05</td>
<td>-1348.15</td>
<td></td>
</tr>
<tr>
<td>49.0</td>
<td>7.350</td>
<td>1387.59</td>
<td>1399.98</td>
<td>-964.97</td>
<td>-1059.55</td>
<td>-1378.15</td>
<td></td>
</tr>
<tr>
<td>50.0</td>
<td>7.500</td>
<td>1415.63</td>
<td>1429.98</td>
<td>-981.87</td>
<td>-1079.05</td>
<td>-1408.15</td>
<td></td>
</tr>
<tr>
<td>GAUSS</td>
<td>X</td>
<td>M\textsubscript{\textit{I}} = 1</td>
<td>M\textsubscript{\textit{I}} = 0</td>
<td>M\textsubscript{\textit{I}} = -1</td>
<td>M\textsubscript{\textit{I}} = -2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>----</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>-1.054,79</td>
<td>-1.054,79</td>
<td>-1.054,79</td>
<td>-1.054,79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20.00</td>
<td>1.436</td>
<td>-1.049,48</td>
<td>-1.049,37</td>
<td>-1.052,10</td>
<td>-1.054,14</td>
<td>-1.071,18</td>
<td></td>
</tr>
<tr>
<td>40.00</td>
<td>2.731</td>
<td>-1.039,80</td>
<td>-1.044,10</td>
<td>-1.073,51</td>
<td>-1.079,27</td>
<td>-1.083,55</td>
<td></td>
</tr>
<tr>
<td>60.00</td>
<td>4.097</td>
<td>-1.026,96</td>
<td>-1.030,85</td>
<td>-1.082,02</td>
<td>-1.089,24</td>
<td>-1.092,66</td>
<td></td>
</tr>
<tr>
<td>80.00</td>
<td>5.443</td>
<td>-1.017,74</td>
<td>-1.026,30</td>
<td>-1.092,15</td>
<td>-1.097,47</td>
<td>-1.111,93</td>
<td></td>
</tr>
<tr>
<td>100.00</td>
<td>6.829</td>
<td>-1.008,56</td>
<td>-1.017,25</td>
<td>-1.110,82</td>
<td>-1.116,81</td>
<td>-1.111,29</td>
<td></td>
</tr>
<tr>
<td>120.00</td>
<td>8.194</td>
<td>-1.003,64</td>
<td>-1.111,32</td>
<td>-1.116,23</td>
<td>-1.120,71</td>
<td></td>
<td></td>
</tr>
<tr>
<td>140.00</td>
<td>9.560</td>
<td>-0.994,57</td>
<td>-1.120,86</td>
<td>-1.125,70</td>
<td>-1.130,18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>160.00</td>
<td>10.812</td>
<td>-0.985,53</td>
<td>-1.130,42</td>
<td>-1.135,21</td>
<td>-1.138,70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>180.00</td>
<td>12.092</td>
<td>-0.976,50</td>
<td>-1.140,10</td>
<td>-1.140,79</td>
<td>-1.141,82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200.00</td>
<td>13.367</td>
<td>-0.967,51</td>
<td>-1.150,89</td>
<td>-1.151,49</td>
<td>-1.151,99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>220.00</td>
<td>14.642</td>
<td>-0.958,54</td>
<td>-1.161,60</td>
<td>-1.162,29</td>
<td>-1.162,79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>240.00</td>
<td>15.919</td>
<td>-0.949,60</td>
<td>-1.172,37</td>
<td>-1.173,06</td>
<td>-1.173,57</td>
<td></td>
<td></td>
</tr>
<tr>
<td>260.00</td>
<td>17.195</td>
<td>-0.940,69</td>
<td>-1.183,17</td>
<td>-1.183,89</td>
<td>-1.183,97</td>
<td></td>
<td></td>
</tr>
<tr>
<td>280.00</td>
<td>18.471</td>
<td>-0.931,78</td>
<td>-1.194,02</td>
<td>-1.194,72</td>
<td>-1.194,72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>300.00</td>
<td>19.748</td>
<td>-0.922,86</td>
<td>-1.204,90</td>
<td>-1.204,69</td>
<td>-1.204,69</td>
<td></td>
<td></td>
</tr>
<tr>
<td>320.00</td>
<td>21.025</td>
<td>-0.914,95</td>
<td>-1.215,82</td>
<td>-1.215,61</td>
<td>-1.215,61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>340.00</td>
<td>22.302</td>
<td>-0.907,04</td>
<td>-1.226,79</td>
<td>-1.226,60</td>
<td>-1.226,60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>360.00</td>
<td>23.579</td>
<td>-0.899,13</td>
<td>-1.237,77</td>
<td>-1.237,67</td>
<td>-1.237,67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>380.00</td>
<td>24.856</td>
<td>-0.891,24</td>
<td>-1.248,80</td>
<td>-1.248,72</td>
<td>-1.248,72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>400.00</td>
<td>26.133</td>
<td>-0.883,37</td>
<td>-1.259,85</td>
<td>-1.259,79</td>
<td>-1.259,79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>420.00</td>
<td>27.409</td>
<td>-0.875,51</td>
<td>-1.270,86</td>
<td>-1.270,80</td>
<td>-1.270,80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The table continues with similar entries for subsequent power values.
<table>
<thead>
<tr>
<th>GAUSS</th>
<th>X</th>
<th>MI=0.5</th>
<th>MI=+-0.5</th>
<th>MI=-0.5</th>
<th>MI=-+-0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>1280</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>1040</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>800</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>4</td>
<td>200</td>
<td>560</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
<td>400</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>6</td>
<td>200</td>
<td>280</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>7</td>
<td>200</td>
<td>200</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
<tr>
<td>8</td>
<td>200</td>
<td>140</td>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
</tr>
</tbody>
</table>

**TABLE VI**

TRITIUM ATOM 2S STATES

<table>
<thead>
<tr>
<th>ALPHA STATES</th>
<th>BETA STATES</th>
</tr>
</thead>
<tbody>
<tr>
<td>400.41</td>
<td>575.45</td>
</tr>
<tr>
<td>468.48</td>
<td>636.91</td>
</tr>
<tr>
<td>402.50</td>
<td>572.02</td>
</tr>
<tr>
<td>384.26</td>
<td>511.86</td>
</tr>
<tr>
<td>356.19</td>
<td>497.93</td>
</tr>
<tr>
<td>328.12</td>
<td>440.41</td>
</tr>
<tr>
<td>300.05</td>
<td>380.00</td>
</tr>
<tr>
<td>271.97</td>
<td>310.00</td>
</tr>
<tr>
<td>243.90</td>
<td>280.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MI=0.5</th>
<th>MI=+-0.5</th>
<th>MI=-0.5</th>
<th>MI=-+-0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
</tbody>
</table>

**TABLE VI**

TRITIUM ATOM 2S STATES

<table>
<thead>
<tr>
<th>ALPHA STATES</th>
<th>BETA STATES</th>
</tr>
</thead>
<tbody>
<tr>
<td>400.41</td>
<td>575.45</td>
</tr>
<tr>
<td>468.48</td>
<td>636.91</td>
</tr>
<tr>
<td>402.50</td>
<td>572.02</td>
</tr>
<tr>
<td>384.26</td>
<td>511.86</td>
</tr>
<tr>
<td>356.19</td>
<td>497.93</td>
</tr>
<tr>
<td>328.12</td>
<td>440.41</td>
</tr>
<tr>
<td>300.05</td>
<td>380.00</td>
</tr>
<tr>
<td>271.97</td>
<td>310.00</td>
</tr>
<tr>
<td>243.90</td>
<td>280.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MI=0.5</th>
<th>MI=+-0.5</th>
<th>MI=-0.5</th>
<th>MI=-+-0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
<tr>
<td>-1517/3049</td>
<td>-1484/549</td>
<td>-1452/222</td>
<td>-1420/232</td>
</tr>
</tbody>
</table>
### TABLE VII

<table>
<thead>
<tr>
<th>TRITIUM ATOM 2P STATES</th>
<th>ALPHA STATES</th>
<th>BETA STATES</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>GAUSS</th>
<th>X</th>
<th>MI=0.5</th>
<th>MI=0.4</th>
<th>MI=0.3</th>
<th>MI=0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.000</td>
<td>1042.25</td>
<td>1042.25</td>
<td>1055.45</td>
</tr>
<tr>
<td>20.0</td>
<td>2948</td>
<td>1032.93</td>
<td>1040.94</td>
<td>1051.63</td>
<td>1106.87</td>
</tr>
<tr>
<td>40.0</td>
<td>4978</td>
<td>1023.64</td>
<td>1037.29</td>
<td>1060.99</td>
<td>1110.67</td>
</tr>
<tr>
<td>60.0</td>
<td>6978</td>
<td>1014.38</td>
<td>1011.89</td>
<td>1076.41</td>
<td>1116.29</td>
</tr>
<tr>
<td>80.0</td>
<td>7978</td>
<td>1005.15</td>
<td>1025.36</td>
<td>1079.46</td>
<td>1123.73</td>
</tr>
<tr>
<td>100.0</td>
<td>10448</td>
<td>995.95</td>
<td>1018.07</td>
<td>1089.35</td>
<td>1130.58</td>
</tr>
<tr>
<td>120.0</td>
<td>1.761</td>
<td>986.79</td>
<td>1010.34</td>
<td>1098.96</td>
<td>1138.66</td>
</tr>
<tr>
<td>140.0</td>
<td>2.055</td>
<td>977.66</td>
<td>1002.31</td>
<td>1104.41</td>
<td>1147.11</td>
</tr>
<tr>
<td>160.0</td>
<td>2.348</td>
<td>968.56</td>
<td>994.06</td>
<td>1117.99</td>
<td>1155.82</td>
</tr>
<tr>
<td>180.0</td>
<td>2.642</td>
<td>959.49</td>
<td>985.75</td>
<td>1127.60</td>
<td>1164.72</td>
</tr>
<tr>
<td>200.0</td>
<td>2.935</td>
<td>950.65</td>
<td>977.26</td>
<td>1137.24</td>
<td>1173.70</td>
</tr>
<tr>
<td>220.0</td>
<td>3.229</td>
<td>941.45</td>
<td>968.75</td>
<td>1146.91</td>
<td>1182.96</td>
</tr>
<tr>
<td>240.0</td>
<td>3.522</td>
<td>932.48</td>
<td>960.14</td>
<td>1156.62</td>
<td>1192.24</td>
</tr>
<tr>
<td>260.0</td>
<td>3.816</td>
<td>923.54</td>
<td>951.61</td>
<td>1166.34</td>
<td>1201.62</td>
</tr>
<tr>
<td>280.0</td>
<td>4.109</td>
<td>914.63</td>
<td>943.02</td>
<td>1176.13</td>
<td>1211.06</td>
</tr>
<tr>
<td>300.0</td>
<td>4.403</td>
<td>905.75</td>
<td>934.42</td>
<td>1185.43</td>
<td>1220.58</td>
</tr>
<tr>
<td>320.0</td>
<td>4.696</td>
<td>896.91</td>
<td>925.82</td>
<td>1195.77</td>
<td>1230.15</td>
</tr>
<tr>
<td>340.0</td>
<td>4.990</td>
<td>888.10</td>
<td>917.24</td>
<td>1206.43</td>
<td>1239.79</td>
</tr>
<tr>
<td>360.0</td>
<td>5.283</td>
<td>879.32</td>
<td>908.66</td>
<td>1217.19</td>
<td>1249.47</td>
</tr>
<tr>
<td>380.0</td>
<td>5.577</td>
<td>870.57</td>
<td>900.09</td>
<td>1227.96</td>
<td>1259.20</td>
</tr>
<tr>
<td>400.0</td>
<td>5.871</td>
<td>861.85</td>
<td>891.55</td>
<td>1238.72</td>
<td>1269.99</td>
</tr>
<tr>
<td>420.0</td>
<td>6.164</td>
<td>853.17</td>
<td>883.02</td>
<td>1249.48</td>
<td>1280.81</td>
</tr>
<tr>
<td>440.0</td>
<td>6.458</td>
<td>844.51</td>
<td>874.51</td>
<td>1259.36</td>
<td>1290.68</td>
</tr>
<tr>
<td>460.0</td>
<td>6.751</td>
<td>835.89</td>
<td>866.02</td>
<td>1269.12</td>
<td>1300.69</td>
</tr>
<tr>
<td>480.0</td>
<td>7.045</td>
<td>827.30</td>
<td>857.56</td>
<td>1278.63</td>
<td>1310.84</td>
</tr>
<tr>
<td>500.0</td>
<td>7.333</td>
<td>818.75</td>
<td>849.12</td>
<td>1288.17</td>
<td>1320.35</td>
</tr>
<tr>
<td>520.0</td>
<td>7.628</td>
<td>810.22</td>
<td>840.71</td>
<td>1297.71</td>
<td>1330.35</td>
</tr>
<tr>
<td>540.0</td>
<td>7.925</td>
<td>801.73</td>
<td>832.32</td>
<td>1307.25</td>
<td>1340.46</td>
</tr>
<tr>
<td>560.0</td>
<td>8.219</td>
<td>793.27</td>
<td>823.96</td>
<td>1316.76</td>
<td>1350.68</td>
</tr>
<tr>
<td>580.0</td>
<td>8.512</td>
<td>784.84</td>
<td>815.62</td>
<td>1326.28</td>
<td>1360.96</td>
</tr>
<tr>
<td>600.0</td>
<td>8.806</td>
<td>776.44</td>
<td>807.31</td>
<td>1335.80</td>
<td>1371.26</td>
</tr>
<tr>
<td>620.0</td>
<td>9.099</td>
<td>768.08</td>
<td>799.03</td>
<td>1345.32</td>
<td>1381.60</td>
</tr>
<tr>
<td>640.0</td>
<td>9.393</td>
<td>759.74</td>
<td>790.78</td>
<td>1354.84</td>
<td>1391.99</td>
</tr>
<tr>
<td>660.0</td>
<td>9.686</td>
<td>751.44</td>
<td>782.56</td>
<td>1364.36</td>
<td>1402.40</td>
</tr>
<tr>
<td>680.0</td>
<td>9.980</td>
<td>743.17</td>
<td>774.36</td>
<td>1373.88</td>
<td>1412.91</td>
</tr>
<tr>
<td>700.0</td>
<td>10.273</td>
<td>734.93</td>
<td>766.20</td>
<td>1383.41</td>
<td>1423.44</td>
</tr>
<tr>
<td>720.0</td>
<td>10.567</td>
<td>726.73</td>
<td>758.06</td>
<td>1392.94</td>
<td>1434.00</td>
</tr>
<tr>
<td>740.0</td>
<td>10.860</td>
<td>718.55</td>
<td>750.95</td>
<td>1402.49</td>
<td>1444.57</td>
</tr>
<tr>
<td>760.0</td>
<td>11.154</td>
<td>710.41</td>
<td>743.87</td>
<td>1412.06</td>
<td>1455.16</td>
</tr>
<tr>
<td>780.0</td>
<td>11.448</td>
<td>702.30</td>
<td>736.82</td>
<td>1421.64</td>
<td>1465.77</td>
</tr>
<tr>
<td>800.0</td>
<td>11.741</td>
<td>694.23</td>
<td>729.80</td>
<td>1431.24</td>
<td>1476.38</td>
</tr>
<tr>
<td>820.0</td>
<td>12.035</td>
<td>686.18</td>
<td>721.82</td>
<td>1440.83</td>
<td>1487.00</td>
</tr>
<tr>
<td>840.0</td>
<td>12.328</td>
<td>678.17</td>
<td>713.86</td>
<td>1449.43</td>
<td>1497.63</td>
</tr>
<tr>
<td>860.0</td>
<td>12.622</td>
<td>670.18</td>
<td>705.91</td>
<td>1458.03</td>
<td>1508.28</td>
</tr>
<tr>
<td>880.0</td>
<td>12.915</td>
<td>662.23</td>
<td>698.04</td>
<td>1466.64</td>
<td>1518.92</td>
</tr>
<tr>
<td>900.0</td>
<td>13.209</td>
<td>654.32</td>
<td>690.16</td>
<td>1475.25</td>
<td>1529.56</td>
</tr>
<tr>
<td>920.0</td>
<td>13.502</td>
<td>646.43</td>
<td>682.32</td>
<td>1483.86</td>
<td>1539.20</td>
</tr>
<tr>
<td>940.0</td>
<td>13.796</td>
<td>638.57</td>
<td>674.50</td>
<td>1492.46</td>
<td>1548.84</td>
</tr>
<tr>
<td>960.0</td>
<td>14.089</td>
<td>630.75</td>
<td>666.74</td>
<td>1501.07</td>
<td>1558.48</td>
</tr>
<tr>
<td>980.0</td>
<td>14.383</td>
<td>622.96</td>
<td>659.00</td>
<td>1509.68</td>
<td>1568.12</td>
</tr>
<tr>
<td>1000.0</td>
<td>14.676</td>
<td>615.20</td>
<td>651.28</td>
<td>1518.30</td>
<td>1577.74</td>
</tr>
</tbody>
</table>
m_p, the field required to induce the transition is parallel to the quantization axis, while if Δm = ±1, it is perpendicular to the quantization axis.

b. Magnetic Dipole Selection Rules
The magnetic dipole operator is of the form -μ·B, where B is usually an oscillating or rotating magnetic field, and where μ may be an electronic or nuclear magnetic moment. This operator can have nonzero matrix elements only between states of the same parity. This means, at least for the present case, Δθ = 0.

For very strong fields we have Δm_I = 0, ±1 and Δθ = Δm_S = Δm_L = Δm_J = 0; or Δm_I = 0, ±1 and Δθ = Δm_S = Δm_J = 0. For intermediate fields we have Δm_I = 0, ±1; Δm_S = 0, ±1 and Δm_L = 0, ±1 and Δθ = Δm_J = 0. For weak fields we have Δm_I = 0, ±1, Δm_S = 0, ±1.

Again the magnitude of Δm determines, in the same way as above, the parallel or perpendicular nature of the field required to induced the transition.

Finally, we note that F = 0 + F = 0 is absolutely forbidden for one-quantum transitions.

4. DISCUSSION OF THE GENERAL SPIN STATE SELECTION PROBLEM
A wide assortment of methods exists which might be used to polarize a metastable hydrogen atomic beam. This is in contrast to the problem of spin state selection in an ordinary (ground state) hydrogen atomic beam where only magnetic dipole transitions between the various hyperfine components, or adiabatic reduction of the magnetic field, may be considered.

It is believed that a "three-level interaction," in which the applied fields may simultaneously cause transitions among three levels, offers the best solution to the selection problem. This technique, which was first demonstrated and explained by Lamb and Retherford, and Lamb, is the method used in the LASL ion source. However, we first consider some of the various other possibilities by which a polarized metastable atomic beam may be produced.

In a magnetic field of about 575 G the θ and e states become degenerate. If a small (a few V/cm) transverse electric field is applied, the θ-states are coupled to the short-lived e-states (the half-life of the e-states is ~1.6 nsec) and decay rapidly.

Thus one can obtain a beam of α metastables. Such a beam is analogous to the beam obtained in conventional polarized ion sources after separation in a quadrupole or sextupole field. That is, there is 100% electronic polarization but no nuclear polarization. Such a beam could then be converted to a partially polarized negative ion beam by adiabatically reducing the magnetic field to near zero before the H^2S + A → H^- + A^+ reaction is allowed to occur. However, these particles have a relatively high velocity (~3 x 10^7 cm/sec) and thus a long and carefully designed decreasing B field region is probably needed. Drake and Krotkov, who first used this method, attributed the fact that they obtained only ~2/3 of the theoretical polarization to an inadequate length in their B field transition region.

To obtain increased polarization, one must turn to a selective transition scheme of some sort. We first consider the use of a magnetic dipole transition between a particular α state and a particular θ state in exactly the fashion often used in conventional polarized ion sources. One finds, however (see Section 6), that the electric dipole matrix elements are of the order of 1/a (=137) times larger than the magnetic dipole matrix elements. Even though a line through an rf cavity can be found where B is maximum and E is zero, for practical beam sizes the average E field will be sufficiently large to make the electric dipole transition rate far exceed the magnetic dipole transition rate.

We ask if these electric dipole transitions might be directly employed for our purposes. For hydrogen or tritium atoms the α-θ frequency separation for the two m_I values is approximately 120 MHz; this is to be compared with the natural width of 100 MHz for the θ (and e) levels. Thus one could possibly obtain a reasonable polarization with such a transverse electric field transition. However, both the beam intensity and polarization depend critically on the rf power level. For deuterium atoms the corresponding frequency separation is only about 18 MHz, so for this case the method is infeasible. The corresponding α-θ (longitudinal electric field) transitions are separated by one half the corresponding α-θ separation; thus these
transitions would be even less favorable.

It has been demonstrated\(^\text{10}\) that at the a-d crossing (\(-2360 \text{ G}\)) a static electric field may be used to preferentially quench a single nuclear spin state. For protons or tritons, only \(a(m_I = \frac{-1}{2})\) and \(d(m_I = \frac{1}{2})\) can be coupled by electric dipole radiation. Since this transition violates the strong field selection rules \(\Delta m_I = 0, \Delta m_J = 0\), it is "first-order forbidden." However, the remaining transitions involve \(\Delta m_I > 1\) and, because they are incompatible with dipole radiation, are more highly forbidden. Figure 4 illustrates this situation for both spin 1/2 and spin 1 nuclei. It is seen that, in the deuteron case, two transitions are compatible with \(\Delta m_I = 1\) (transverse electric field) and one with \(\Delta m_I = 0\) (longitudinal electric field). Thus, one could selectively quench one magnetic substate for spin 1/2 systems and either one or two magnetic substates for spin 1 systems. Since the transitions here are first-order forbidden, relatively large electric fields are needed and serious loss will occur through the a-f or a-e transitions. It has been estimated\(^\text{10}\) that, for protons, one might obtain 50% polarization with 25% a state survival.

Another proposal\(^\text{11}\) involves the use of a radio-frequency transition at zero magnetic field. The \(2S_{1/2}(F = 1)\) level may be coupled to the \(2P_{1/2}(F = 0)\) level by longitudinal or transverse radiation of the appropriate frequency. A small magnetic field is permissible if both transverse and longitudinal radiations are present. The \(2S_{1/2}(F = 0)\) state is not appreciably quenched by coupling to the \(2P_{1/2}(F = 1)\) state because the frequency difference is \(\approx 236 \text{ MHz}\); the \(2S_{1/2}(F = 0)\) to \(2P_{1/2}(F = 0)\) transition is strictly forbidden. Thus, one might obtain a pure \(2S_{1/2}(F = 0)\) metastable beam. If the field were then adiabatically increased to a high value, the metastable beam would have 100% nuclear and 100% electronic longitudinal polarization. This scheme, however, is applicable only to protons or tritons.

5. QUANTUM MECHANICAL FORMULATION OF THE FOUR-LEVEL PROBLEM

We consider only the four-level system \(a, b, e,\) and \(f\) since the \(2P_{3/2}\) levels are sufficiently distant to have no significant effect on our problem. In addition, since we are working in a strong magnetic field, the nuclear magnetic quantum number \(m_I\) is conserved and we may therefore consider separately each group of four atomic levels associated with a particular nuclear spin orientation.

For the amplitudes of the \(a, b, e,\) and \(f\) states, we use the notation \(a, b, c,\) and \(d\). We use \(\omega\) (with or without subscripts) to denote an angular frequency.

The Schrödinger equation may be written:

\[
(H_0 + H')\psi = \hbar\frac{\partial \psi}{\partial t},
\]

where \(H_0\) is a time-independent Hamiltonian whose eigenfunctions satisfy the equation \(H_0 u_n = E_n u_n\).

If the exact wave function is written in the form

\[
\psi = \sum a_n(t) u_n e^{-i\omega_n t},
\]

it is easy to show that the coefficients \(a_n(t)\) must satisfy the differential equations

\[
if\phi_k = i H_0 a_n e^{-i\omega_n t},
\]

where

\[
u_{kn} = (E_k - E_n) / \hbar,
\]
For the four-level case, these equations may be written out explicitly as:

\[
\begin{bmatrix}
  i\hbar & 0 & i\omega_{ab} & i\omega_{df} \\
  i\hbar & i\omega_{ba} & 0 & i\omega_{bc} \\
  i\hbar & i\omega_{ca} & i\omega_{ae} & -\gamma/2 \\
  i\hbar & i\omega_{db} & i\omega_{fd} & i\omega_{ef} \\
\end{bmatrix}
\]

where the damping terms \(-i\gamma/2\) and \(-i\gamma/2\) have been added to account for the decay of the e and f states. Except for the damping terms, the above matrix is Hermitian. The a-β and e-γ transitions may be induced only by a transverse magnetic field; the a-γ and β-γ transitions may be induced only by a transverse electric field; and the a-δ and β-δ transitions may be induced only by a longitudinal electric field.

We assume a longitudinal oscillating electric field (angular frequency \(\omega\)), with an associated transverse oscillating magnetic field, and a transverse static electric field. The matrix elements may then be written in the form

\[
\begin{align*}
H'_{ba} &= i\hbar \cos \omega t \\
H'_{fe} &= i\hbar \cos \omega t \\
\end{align*}
\]

where \(M\) and \(M'\) represent magnetic dipole matrix elements, \(R\) and \(R'\) represent longitudinal electric dipole matrix elements, and \(V\) and \(V'\) represent transverse electric dipole matrix elements. (The notation \(R\) and \(V\) is selected since \(R\) and \(R'\) will be associated with a radio-frequency electric field, and \(V\) and \(V'\) will be associated with a static electric field.) These matrix elements are discussed in Section 6; for the present we merely observe that the magnitude of dipole matrix elements is directly proportional to the relevant applied field. Note that the unprimed matrix elements relate to the e level, and the primed matrix elements relate to the f level.

In this notation our equations become:

\[
\begin{bmatrix}
i\hbar & 0 & i\omega_{ab} & i\omega_{df} \\
i\hbar & i\omega_{ba} & 0 & i\omega_{bc} \\
i\hbar & i\omega_{ca} & i\omega_{ae} & -\gamma/2 \\
i\hbar & i\omega_{db} & i\omega_{fd} & i\omega_{ef} \\
\end{bmatrix}
\]

\[
\begin{align*}
H'_{ba} &= fM' \cos \omega t \\
H'_{fe} &= fM' \cos \omega t \\
\end{align*}
\]
The equations are used in this form for the numerical integration studies discussed in Sections 8 and 9. However, the effect of the f state on the system is not large—it merely causes a slow overall decay of the metastable beam. For our first discussion of the system, we neglect the f level effects.

Also, except in the numerical integration studies, we follow the standard practice and drop the term in cos wt = ½ (e^iwt + e^-iwt) which is incapable of resonance. (We note that this approximation cannot be made for both frequencies if one uses an oscillating transverse electric field as well as an oscillating longitudinal electric field, since then at least two frequency terms can always resonate. In fact, one then obtains interference between the two contributions.) With these approximations the equations become:

\[
\begin{bmatrix}
  i(\omega_0 - \omega)t \\
  -i(\omega_0 + \omega)t \\
  -i(\omega_0 - \omega)t \\
  -i(\omega_0 + \omega)t
\end{bmatrix}
\begin{bmatrix}
  i(\omega_0 - \omega)t \\
  R e^{i\omega_0 t} \\
  -i\gamma/2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  e^{-i\omega_0 t} \\
  e^{-i\omega_0 t} \\
  -1/2
\end{bmatrix}\]

These are the equations given by Lamb. In Section 7 we follow the method indicated by Lamb to obtain an analytic solution to these equations for the special case of constant field magnitudes during the interaction time.

6. MATRIX ELEMENTS

Neglecting the nuclear hyperfine interaction, the wave functions which describe the n = 2 states of the hydrogen atom may be written in the form given in Table VIII. The coefficients \(\xi_1 - \xi_4\) can be expressed as follows. Define the dimensionless parameter \(\xi = u_0 B/AE\), where \(AE\) is the fine structure splitting (10,968 MHz). (\(\xi\) becomes unity at a field of about 7800 G; thus, as far as fine structure is concerned, we are interested primarily in the weak field region. Accordingly, Table VIII is subdivided into the weak field groups (where \(J\) and \(m_J\) are good quantum numbers) although the wave functions are exact for all fields.) We may write

\[
\xi_1 = \frac{\sqrt{3}}{4}(1 + \delta_+),
\]

\[
\xi_2 = \frac{\sqrt{3}}{2}(1 - \delta_+),
\]

\[
\xi_3 = \frac{\sqrt{3}}{2}(1 + \delta_-),
\]

\[
\xi_4 = \frac{\sqrt{3}}{4}(1 - \delta_-),
\]

where

\[
\delta_+ = (\xi + 1/3)/\sqrt{1 + 2\xi/3 + \xi^2}
\]

and

\[
\delta_- = (\xi - 1/3)/\sqrt{1 - 2\xi/3 + \xi^2}
\]

The quantities \(\xi, \delta_+, \delta_-, \) and \(\xi_1 - \xi_4\) are tabulated, for various magnetic fields, in Table IX. We note that, for zero field, \(\xi_+ = 1/3, \xi_- = -1/3\) and \(\xi_1 - \xi_4\) become unity. In that case, the coefficients in Table IX become the usual Clebsch-Gordan coefficients which couple angular momenta 1 and 1/2. For large fields, \(\xi_+ = 1\) and \(\xi_- = 1\); thus \(\xi_2\) and \(\xi_3\) become zero while \(\xi_1 = \sqrt{3}/2\) and \(\xi_3 = \sqrt{3}/2\). In this case, we obtain the wave functions for which \(l, m_1, s,\) and \(m_s\) are the appropriate quantum numbers.

We now consider the effect of the nuclear hyperfine

* The values of \(\xi_1 - \xi_4\) were obtained from Bethe and Salpeter, Section 46. The Clebsch-Gordan coefficients and angular functions used throughout this section are, however, those of Condon and Shortley. The tables of matrix elements given in Ref. 13 may be used if account is taken of the \((-1)^M\) difference in phase conventions for the spherical harmonics.
**TABLE VIII**  
*n* = 2 Hydrogen Atom Wave Functions in a Magnetic Field

<table>
<thead>
<tr>
<th>Multiplet</th>
<th>State</th>
<th>( m_j )</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>3/2</td>
<td>( R_{21} \psi_{11} ) (+)</td>
</tr>
<tr>
<td>2P&lt;sub&gt;3/2&lt;/sub&gt;</td>
<td>b</td>
<td>1/2</td>
<td>( \frac{\sqrt{2}}{3} R_{21} \psi_{10} ) (+) + ( \frac{\sqrt{3}}{3} R_{21} \psi_{11} ) (+)</td>
</tr>
<tr>
<td></td>
<td>c</td>
<td>-1/2</td>
<td>( \frac{\sqrt{2}}{3} R_{21} \psi_{1-1} ) (+) + ( \frac{\sqrt{3}}{3} R_{21} \psi_{10} ) (+)</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>-3/2</td>
<td>( R_{21} \psi_{1-1} ) (+)</td>
</tr>
<tr>
<td>2P&lt;sub&gt;1/2&lt;/sub&gt;</td>
<td>e</td>
<td>1/2</td>
<td>( \frac{\sqrt{2}}{3} R_{21} \psi_{10} ) (+) + ( \frac{\sqrt{3}}{3} R_{21} \psi_{11} ) (+)</td>
</tr>
<tr>
<td></td>
<td>f</td>
<td>-1/2</td>
<td>( \frac{\sqrt{2}}{3} R_{21} \psi_{1-1} ) (+) + ( \frac{\sqrt{3}}{3} R_{21} \psi_{10} ) (+)</td>
</tr>
<tr>
<td>2S&lt;sub&gt;1/2&lt;/sub&gt;</td>
<td>a</td>
<td>1/2</td>
<td>( R_{20} \psi_{00} ) (+)</td>
</tr>
<tr>
<td></td>
<td>b</td>
<td>-1/2</td>
<td>( R_{20} \psi_{00} ) (+)</td>
</tr>
</tbody>
</table>

**NOTATION**

(+) and (+) are electron spinors

\[
R_{20} = \frac{1}{\sqrt{2}} e^{-i \theta} (1 - e^{i \phi})
\]

\[
R_{21} = \frac{1}{2 \sqrt{3}} e^{-i \theta}
\]

\[
\psi_{11} = -\frac{3}{2} \sqrt{\frac{2}{4\pi}} \sin \theta e^{-i \phi}
\]

\[
\psi_{10} = \frac{3}{4\pi} \cos \theta
\]

\[
\psi_{1-1} = -\frac{3}{8\pi} \sin \theta e^{-i \phi}
\]

\[
\psi_{00} = \frac{3}{8\pi}
\]
TABLE IX
Factors Required for the Computation of the Hydrogen Atom Wave Functions
in an Arbitrary Magnetic Field

<table>
<thead>
<tr>
<th>Field (G)</th>
<th>ξ</th>
<th>δ_+</th>
<th>δ_-</th>
<th>ε₁</th>
<th>ε₂</th>
<th>ε₃</th>
<th>ε₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>.3333</td>
<td>- .3333</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>390</td>
<td>.05</td>
<td>.3833</td>
<td>- .2833</td>
<td>1.016</td>
<td>.983</td>
<td>1.034</td>
<td>.967</td>
</tr>
<tr>
<td>780</td>
<td>.1</td>
<td>.4333</td>
<td>- .2333</td>
<td>1.031</td>
<td>.965</td>
<td>1.068</td>
<td>.935</td>
</tr>
<tr>
<td>3,900</td>
<td>.5</td>
<td>.8333</td>
<td>- .1666</td>
<td>1.117</td>
<td>.872</td>
<td>1.113</td>
<td>.712</td>
</tr>
<tr>
<td>7,800</td>
<td>1.0</td>
<td>1.3333</td>
<td>+ .6666</td>
<td>1.167</td>
<td>.563</td>
<td>1.538</td>
<td>.525</td>
</tr>
<tr>
<td>39,000</td>
<td>5.0</td>
<td>5.3333</td>
<td>+ .6666</td>
<td>1.220</td>
<td>.122</td>
<td>1.724</td>
<td>.152</td>
</tr>
<tr>
<td>78,000</td>
<td>10.0</td>
<td>10.3333</td>
<td>+ 9.6666</td>
<td>1.224</td>
<td>.084</td>
<td>1.730</td>
<td>.111</td>
</tr>
</tbody>
</table>

interaction. The magnetic fields of interest are in the region of 500-600 G. The parameter which defines the strong and weak field regions (with respect to the hyperfine interaction) is \( X = \frac{\nu \mu_B B}{\Delta \omega} \), where \( \Delta \omega \) corresponds to the zero field hyperfine splitting. The magnetic field at which \( X \) is unity varies from \( \sim 15 \) to \( \sim 65 \) G for the cases of present interest (see Table I) and we are thus interested in the strong field region as far as the hyperfine interaction is concerned. This means that the appropriate wave functions which include the nuclear spin are of the form given in Table VIII multiplied by a nuclear spin wave function which corresponds to a particular \( m_s \). (These are only approximately correct wave functions; in fact, a small amount of nuclear spin component of other than the predominant \( m_s \) value will be present. The situation is analogous to the situation discussed above for the fine structure wave functions, if \( \tilde{F}, \tilde{J}, \) and \( \tilde{F} \) replace \( \tilde{J}, \tilde{L}, \) and \( \tilde{S} \). The \( \alpha-\beta \) crossing technique discussed in Section 4 makes use of this fact.)

Turning to the evaluation of the matrix elements which connect the various states, we assume that \( m_\perp \) is a good quantum number; i.e., that we may use electron wave functions of the form given in Table VIII multiplied by a nuclear spinor. The selection rule \( \Delta m_\perp = 0 \) holds for the transitions of interest so the nuclear spin wave function is omitted in the following discussion. Further, we assume that the direction of the static magnetic field defines the \(+z\) axis of the system.

We first consider the magnetic dipole matrix elements which connect the \( \beta \) to the \( \alpha \) and the \( \beta \) to the \( \epsilon \) states. The perturbing interaction is

\[
H' = - \vec{B} \cdot \vec{\beta}' = -g_\beta \mu_B \vec{L} \cdot \vec{B}' - g_s \mu_s \vec{S} \cdot \vec{B}'
\]

where, neglecting radiative corrections, \( g_\beta = 1 \) and \( g_s = 2 \), and where \( \mu_s \) is the Bohr magneton. We may write:

\[
\vec{L} \cdot \vec{B}' = \mathfrak{I}_{B}(B_x + i B_y)(\mathfrak{I}_x - i \mathfrak{I}_y) + \mathfrak{I}_{B'}(B_x - i B_y)
\]

Using this expression together with the similar expression for \( \hat{S} \cdot \vec{B}' \), the \( \beta-\alpha \) matrix element may be written:

\[
\langle \beta \alpha | H' | \beta \alpha \rangle = -\frac{\nu \mu_B}{2} \int R_{20} \psi_{00}(t) \left( \mathfrak{I}_{B x} (g_\beta \mathfrak{I}_x + g_s \mathfrak{S}_x) + \mathfrak{I}_{B y} (g_\beta \mathfrak{I}_y + g_s \mathfrak{S}_y) \right) dt
\]

The operators \( \mathfrak{I}_x, \mathfrak{I}_y, \) and \( \mathfrak{I}_z \) obey the equations:

\[
\begin{align*}
\mathfrak{I}_+ \psi_{m_k} = & \sqrt{(k+ m_k)(- k+ m_k + 1)} \psi_{m_k} + 1 \\
\mathfrak{I}_- \psi_{m_k} = & \sqrt{(k+ m_k)(- k+ m_k + 1)} \psi_{m_k} - 1, \quad \text{and} \\
\mathfrak{I}_z \psi_{m_k} = & m_k \psi_{m_k}
\end{align*}
\]

The operators \( \mathfrak{s}_+, \mathfrak{s}_-, \) and \( \mathfrak{s}_z \) obey identical equa-
### TABLE X

#### n = 2 Electric Dipole Matrix Elements

<table>
<thead>
<tr>
<th>Transition</th>
<th>Δm</th>
<th>Matrix Element of $e\mathbf{E} \cdot \mathbf{r}$</th>
<th>Matrix Element of $y\mathbf{E}_x$</th>
<th>Matrix Element of $z\mathbf{E}_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>α + α</td>
<td>+1</td>
<td>$-3e_2/\sqrt{2}$</td>
<td>$-3e_2/\sqrt{2}$</td>
<td>0</td>
</tr>
<tr>
<td>β + d</td>
<td>-1</td>
<td>$3e_2/\sqrt{2}$</td>
<td>$3e_2/\sqrt{2}$</td>
<td>0</td>
</tr>
<tr>
<td>α + b</td>
<td>0</td>
<td>$\sqrt{6}e_2E_z$</td>
<td>0</td>
<td>$\sqrt{6}e_2E_z$</td>
</tr>
<tr>
<td>β + c</td>
<td>0</td>
<td>$\sqrt{6}e_2E_z$</td>
<td>0</td>
<td>$\sqrt{6}e_2E_z$</td>
</tr>
<tr>
<td>α + c</td>
<td>-1</td>
<td>$\sqrt{3}/2e_2E_+^r$</td>
<td>$\sqrt{3}/2e_2E_x^r$</td>
<td>0</td>
</tr>
<tr>
<td>β + b</td>
<td>+1</td>
<td>$-\sqrt{3}/2e_2E_-^r$</td>
<td>$-\sqrt{3}/2e_2E_x^r$</td>
<td>0</td>
</tr>
<tr>
<td>α + d</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>β + a</td>
<td>+2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>α + e</td>
<td>0</td>
<td>$-\sqrt{3}e_2E_z$</td>
<td>0</td>
<td>$-\sqrt{3}e_2E_z$</td>
</tr>
<tr>
<td>β + f</td>
<td>0</td>
<td>$\sqrt{3}e_2E_z$</td>
<td>0</td>
<td>$\sqrt{3}e_2E_z$</td>
</tr>
<tr>
<td>α + f</td>
<td>-1</td>
<td>$-\sqrt{3}e_2E_z$</td>
<td>$-\sqrt{3}e_2E_x$</td>
<td>0</td>
</tr>
<tr>
<td>β + e</td>
<td>+1</td>
<td>$-\sqrt{3}e_2E_z$</td>
<td>$\sqrt{3}e_2E_x$</td>
<td>0</td>
</tr>
</tbody>
</table>

For the present example, the $\mathbf{k}_+$, $\mathbf{k}_-$, and $\mathbf{k}_z$ operators yield zero and the only contribution to the integral is from the $\mathbf{k}_zB_+^rS_+^r$ term. We thus obtain $H'_{\mathbf{k}z} = -\sqrt{g_sB_+^rS_+^r}$, which is the notation of Section 5 this corresponds to $M' = -\mu_0B_x^r/\hbar$ or $M' = -ieB_y^r/\hbar$ for oscillating fields in the $x$ and $y$ directions, respectively.

The matrix element connecting $f$ to $e$ can be similarly evaluated; the result is:

$$H'_{\mathbf{k}z} = -\mu_0B_x^r/3 .$$

(This limiting result could have been readily obtained by considering the effective interaction to be $-g_2\mu_0\mathbf{E} \cdot \mathbf{r}$ together with the $Jm_j$ representation of the state.) In the notation of Section 5, this result corresponds to $M' = -ie\mathbf{B}_{\mathbf{x}}/\hbar$ or $M' = -ie\mathbf{B}_{\mathbf{y}}/\hbar$ for oscillating fields in the $x$ and $y$ directions, respectively, where $s$ is given by:

$$s = g_1(\epsilon_1\epsilon_4 + \epsilon_2\epsilon_3)/3 + g_9\epsilon_2\epsilon_3/2 .$$

We note that $s = 1$ for field strengths of present interest. Numerically the quantity $\mu_0/\hbar$ is $2\pi$ times $(1.401)\ MHz/G$. Note that $M$ and $M'$ have units of angular frequency.

We now turn to the electric dipole matrix elements. In this case the perturbing part of the Hamiltonian is of the form $H' = e\mathbf{E} \cdot \mathbf{r}$ where $e$ is the electronic charge, $\mathbf{E}$ an electric field strength, and $\mathbf{r}$ is the electron position vector.

For example, the matrix element which causes the
transition $a \rightarrow e$ may be written

$$H'_{ae} = \int \{ e_2^R_2 R^9_19(t) + e_1^R_2 R^9_11(t) \}[e^2 \cdot \mathbf{F}]$$

$$R^{9_00}(t) dt .$$

Noting the orthonormality of the electron spinors and expanding $e^2 \cdot \mathbf{F}$, this becomes

$$H'_{ae} = e_2^R_2 \int R^9_2 \psi_1^{9_0} 1^2 E_{x-} + 1^2 E_{y+} + E_z] R^{9_00} \psi_0 \psi_1^{9_0} dt .$$

If we use the following facts

$$z = r \cos \theta$$
$$x_+ = r \sin \theta e^{i \phi}$$
$$x_+ = r \sin \theta e^{-i \phi}$$
$$dt = r^2 \sin \theta d\theta d\phi ,$$

we obtain

$$H'_{ae} = e_2^R_2 \int R^9_2 \int \int R^{9_00} \psi_1^{9_0} (E_x \cos \theta + \frac{1}{2} E_y \sin \theta e^{-i \phi} + \frac{1}{2} E_y \sin \theta e^{i \phi}) \psi_0 \sin \theta d\phi d\theta .$$

The radial integral has the value $3 \sqrt{3} a_0$, where $a_0$ is the Bohr radius. The angular integrals can be evaluated either directly or by reference to tables. In this case, only $E_x \cos \theta$ survives the $\phi$ integration, and we obtain

$$H'_{ae} = -3 \sqrt{3} e o \psi_0 \psi_z .$$

In Table X the matrix elements are given with the assumption of the Condon-Shortley conventions for the vector coupling coefficients and wave functions. Any modification of sign in which $e_1$ and $e_2$ and $e_3$ are changed simultaneously, or in which the overall phase of a wave function is changed, preserves the orthogonality and eigenvalues of the functions and is therefore acceptable. Thus, for the $2S_{1/2} \rightarrow 2P_{1/2}$ transitions, many possible consistent sets of signs for the matrix elements are possible.

In the notation of Section 5, the electric dipole matrix elements may be written

$$R = -3 e_2 \psi_0 \psi_z$$
$$R' = \sqrt{3} e_3 \psi_0 \psi_z ,$$

and

$$V = -\sqrt{3} e_2 e o \psi_x \text{ or } -\sqrt{3} e_2 e o \psi_y$$
$$V' = -\sqrt{3} e_3 e o \psi_x \text{ or } \sqrt{3} e_3 e o \psi_y ,$$

depending on whether the transverse field is along the $x$ or $y$ axis. For the magnetic fields of interest, $e_1 - e_3$ differ from unity by only a few percent. Neglecting these small differences, we may write

$$R' = R^* \text{ and } V' = V^* .$$

This form is used in some of the later discussion. In fact, any relative signs between the matrix elements such that $\arg(R/V) = -\arg(R'/V')$ will result in identical answers for any calculation which involves only these matrix elements.

Numerically, the quantity $\sqrt{3} e o \psi_0 / \hbar$ is given by $\sqrt{3} \times (1.60206 \times 10^{-19} \text{ C}) \times (5.29172 \times 10^{-9} \text{ cm}) \times 10^7 / (1.05443 \times 10^{-27} \text{ erg-sec}) = 13.9257 (\text{cm/} \text{V}) \text{ MHz}$. Note that the quantity $\sqrt{3} e o \psi_0 / \hbar$, where $E$ is an electric field strength (V/cm), has the units of angular frequency.

7. ANALYTIC SOLUTION OF THE THREE-LEVEL PROBLEM

For the case of interest, where the magnetic field is such that the $\beta$ and $e$ levels are nearly degenerate, the $\ell$ level has little effect on the system. Thus, to good approximation, we may neglect its presence. (The quality of this approximation will be examined in Section 8.)

The equations which characterize the three-level system are then

$$i \dot{a} = \frac{i \delta + \omega_e}{2} a + \frac{i \delta - \omega_e}{2} e$$
$$i \dot{a} = \frac{i \delta - \omega_e}{2} a + \frac{i \delta + \omega_e}{2} e$$

where we have defined the angular frequency difference $\delta = \omega_{\beta} - \omega$.

Let us first consider the easily-solved special case which corresponds to $\delta = 0$ and $v_{\beta e} = 0$. In other words, we assume a magnetic field strength such that the $\beta$ and $e$ levels are degenerate (crossing) and an rf frequency such that $\omega/2\pi = \omega_{\beta e}/2\pi$ (resonance). Note that we are speaking of a particular nuclear spin magnetic quantum number, since
simultaneous resonance and crossing occur at a different magnetic field (and corresponding frequency) for the various $m_1$ values. We also neglect $M$ (for the reasons given in Section 6).

With these assumptions the equations become

\[ 1 = \frac{\hbar R}{\nu} \frac{\partial c}{\partial t} \]
\[ 1 = \frac{\nu}{\hbar} \frac{\partial c}{\partial t} \]
\[ 1 = \frac{\hbar R + \nu}{\hbar} (1 + \nu c) \]

If one differentiates the third of these equations and substitutes the first two equations into the third, the result is

\[ c + \nu c + P_0 c = 0, \]

where $P_0 = \frac{\hbar R}{\nu} + \nu c_0$. The general solution of this equation is

\[ c = C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t}, \]

where $\mu_1$ and $\mu_2$ are the two roots of $\nu^2 - \nu c + P_0^2 = 0$.

\[ \mu_1, 2 = \frac{\nu}{2} \pm \sqrt{\left(\frac{\nu}{2}\right)^2 - P_0^2}. \]

To evaluate the constants, we assume some initial conditions. For a particle initially in its $\alpha$-state, $a = 1$ and $b = c = 0$ at $t = 0$; this implies that $c = \frac{\nu}{2} \nu R$ at $t = 0$. Applying these conditions, we obtain:

\[ c = \frac{\nu}{2} \nu R \left( e^{-\mu_1 t} - e^{-\mu_2 t} \right), \]

where $\nu = \sqrt{\left(\frac{\nu}{2}\right)^2 - P_0^2}$. This solution is valid for all values of $P_0^2$ except the critically damped case $P_0^2 = \left(\frac{\nu}{2}\right)^2$; for this case $c = \frac{\nu}{2} \nu R \left( e^{-\mu_1 t} - e^{-\mu_2 t} \right)$. The solutions for $c$ may be put back into the equations for $a$ and $b$ to obtain

\[ a = A_3 - \frac{\nu R}{\nu} \int c dt \]
\[ b = B_3 - \frac{\nu R}{\nu} \int c dt, \]

where $A_3$ and $B_3$ are integration constants. We obtain:

\[ a = 1 - \frac{\nu R}{\nu} \frac{\partial c}{\partial t} + \frac{\nu R}{\nu} \left( e^{-\mu_1 t} - e^{-\mu_2 t} \right) \]

After a sufficiently long time the exponential terms decay to zero, since the real parts of $\mu_1$ and $\mu_2$ are positive for all values of $P_0^2$.

Thus, our asymptotic solutions are

\[ a = \frac{\nu R}{\nu} \frac{\partial c}{\partial t} \]
\[ b = -\frac{\nu R}{\nu} \frac{\partial c}{\partial t} \]
\[ c = 0. \]

That is, an equilibrium population of the $\alpha$ and $\beta$ states is established. Since we are dealing with amplitudes, a definite phase relation exists between $a$ and $b$; i.e., we have a coherent mixture of the $\alpha$ and $\beta$ states, while the amplitude for the $\gamma$ state has decayed to zero.

We note that our asymptotic solutions satisfy the condition $\frac{\nu R}{\nu} + \nu R = 0$. From inspection of the equations, it is clear that we have a solution if $c = \frac{\nu}{2} \nu R = 0$. The physical nature of the phenomenon is one of interference; the relative phase of the transition matrix elements is such that contributions from $a$ and $b$ to the $\gamma$ state population destructively interfere, i.e., $\frac{\nu R}{\nu} + \nu R = 0$.

We now turn to the solution of the general three-level equations following the method indicated by Lamb. First, let us generalize the equations slightly to allow for an arbitrary phase for the rf field at $t = 0$. That is, we assume

\[ \frac{\nu R}{\nu} \frac{\partial c}{\partial t} = \frac{\nu R}{\nu} \cos(\omega t + \delta_0), \]

where $\delta_0$ is the phase at $t = 0$. This may be written as

\[ \frac{1}{2} \left( \nu R \frac{\nu}{2} \cos(\omega t + \delta_0) - \nu R \frac{\nu}{2} \cos(\omega t - \delta_0) - \nu R \frac{\nu}{2} \cos(\omega t) \right). \]

We may perform a similar decomposition of $M$. Dropping the negative frequency term, as before, and defining $\Re R = R_0$ and $\Re M = M_0$, the equations remain the same except $R + R_0$ and $M + M_0$. 

23.
Following Lamb, we assume a solution of the form

\[ a = A_1 e^{\nu_1 t} + A_2 e^{\nu_2 t} + A_3 e^{\nu_3 t} \]

\[ b = B_1 e^{\nu_1 t} + B_2 e^{\nu_2 t} + B_3 e^{\nu_3 t} \]

\[ c = (C_1 e^{\nu_1 t} + C_2 e^{\nu_2 t} + C_3 e^{\nu_3 t}) e^{-i\theta t} \]

Substituting this form into the equations and equating coefficients of \( e^{-\nu_1 t} \), for example, we obtain

\[
\begin{bmatrix}
M_0 & i\nu_1 - \delta & V^* \\
R_0 & V & i\nu_1 - \delta - \omega_{Be} - 3i\nu \\
R_0 & V & i\nu_1 - \delta - \omega_{Be} - 3i\nu
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_1 \\
C_1
\end{bmatrix} = 0,
\]

with identical equations holding for \( \nu_2 \) and \( \nu_3 \). We use the general subscript \( k \) from here on since the following discussion applies to \( \nu_1, \nu_2, \) and \( \nu_3 \). For any but the trivial solution \( A_k = B_k = C_k = 0 \), the determinant of the coefficients must vanish; thus, the three values of \( \mu \) are the roots of the complex cubic equation:

\[
i\mu_k (i\mu_k - \delta)(i\mu_k - \delta - \omega_{Be} - 3i\nu) \]

\[
+ (M_0^* R_0 + M_0 V^*) - R_0 R_0^*(i\mu_k - \delta) \]

\[ - V^* (i\mu_k) - M_0 M_0^*(i\mu_k - \delta - \omega_{Be} - 3i\nu) = 0.\]

This may be written in the form

\[ \mu_k^3 + P_k \mu_k^2 + Q_k \mu_k + R = 0, \]

where

\[ P = -i(\Delta_0 + \Delta_3) \]

\[ Q = \nu^* \nu + i(R_{0R_0} + M_{0V^*}) - \Delta_0 \Delta_3 \]

\[ R = i(M_0^* R_0 + M_0 V^*) - k(R_{0R_0} \Delta_3 + M_{0V^*} \Delta_3), \]

and where \( \Delta_0 = -\delta \) and \( \Delta_3 = -\delta - \omega_{Be} - 3i\nu \).

Such a complex cubic equation may be solved algebraically as follows. Define \( \mu_k = z + P/3 \). The equation for \( z \) is then \( z^3 + az + b \) where \( a = Q - P^2/3 \) and \( b = 2P^3/27 - PQ/3 + R \). Then a solution for \( z \) is of the form \( z = u - a/3u \), where \( u \) satisfies the equation \( u^3 = -b/2 \pm \sqrt{(b/2)^2 + (a/3)^2} \). All the operations defined are valid for complex numbers; however, we find six values for \( u \) of which three lead to redundant solutions. To improve computation precision, we select the + or - sign in the equation for \( u^3 \) according to which gives the larger absolute value. It was found necessary to use double-precision arithmetic to achieve satisfactory accuracy for the values of the coefficients of interest to the present problem. The FORTRAN IV code for this procedure is included in Appendix A.

Returning to the matrix equation, once we know that the determinant of the coefficients is zero, we may use any two equations to relate the quantities \( B_k \) and \( C_k \) to \( A_k \), which we will assume to be arbitrary. (That is, \( A_1, A_2, \) and \( A_3 \) will be taken to be the three independent constants characteristic of the solution of a system of three first-order differential equations.) One finds

\[ B_k = -iA_k[M_0 (i\mu_k - \delta - \omega_{Be} - 3i\nu) - R_0 V^*]/D_k, \]

\[ C_k = -iA_k[R_0 (i\mu_k - \delta) - M_0 V]/D_k, \]

where

\[ D_k = (i\mu_k - \delta - \omega_{Be} - 3i\nu)(i\mu_k - \delta) - \nu^* \nu. \]

Thus, defining \( B_k = \epsilon_k A_k \) and \( C_k = \delta_k A_k \), our general solution of the equations is of the form

\[ \begin{bmatrix}
1 & 1 & 1 & \epsilon_k e^{-\nu_1 t} \\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_k e^{-\nu_2 t} \\
\delta_1 & \delta_2 & \delta_3 & \delta_k e^{-\nu_3 t}
\end{bmatrix} \]

where
To evaluate the coefficients $A_1$, $A_2$, and $A_3$, it is necessary to assume some initial conditions. If there is no $e$-state component in the initial beam, we may achieve sufficient generality by assuming each of the initial conditions $a = 1$, $b = 0$, $c = 0$ and $a = 0$, $b = 1$, $c = 0$. The solution to the problem corresponding to a beam which contains an incoherent mixture of $a$-state atoms and $b$-state atoms can then be written by combining these solutions appropriately (i.e., by an average over initial states). We could use some other set of spinors as a basis system; a natural basis system for this problem will be discussed later. For $a = 1$, $b = 0$, and $c = 0$ at $t = 0$, the solution of the linear equations yields

$$A_1 = \frac{(c_2 \delta_3 - c_3 \delta_2)}{D}$$
$$A_2 = \frac{(c_3 \delta_1 - c_1 \delta_3)}{D}$$
$$A_3 = \frac{(c_1 \delta_2 - c_2 \delta_1)}{D},$$

where

$$D = (c_2 \delta_3 - c_3 \delta_2) + (c_3 \delta_1 - c_1 \delta_3) + (c_1 \delta_2 - c_2 \delta_1).$$

For $a = 0$, $b = 1$, and $c = 0$ at $t = 0$ we obtain

$$A_1 = \frac{(\delta_2 - \delta_3)}{D}$$
$$A_2 = \frac{(\delta_3 - \delta_1)}{D}$$
$$A_3 = \frac{(\delta_1 - \delta_2)}{D}.$$
Fig. 6. The variation of $|B_3|$ (solid curves) and arg $B_3$ (dashed curves) for cases 1, 2, and 3. One of the $|B_3|$ curves is terminated at 8.5 G, and a second curve at -8 G, where Re ($\mu_2$) becomes smaller than Re ($\mu_3$).

Fig. 7. The variation of the real part (solid curves) and imaginary part (dashed curves) of $\mu_3$ (in MHz angular frequency) for cases 1, 2, and 3.

substates. However, the values of $\omega/2\pi$ and $B_o$ given above are specifically for $m_1 = 0$ deuterium atoms. Since, for a given fixed frequency, $\omega_2$ will be different for the different substates of the species being polarized, the line shape corresponding to each will be slightly different. The cases 1 and 3 chosen for illustration are probably a little too far from crossing for reasonable separation of deuterium magnetic substates; a range of +35 G from the crossing field would appear to be

Fig. 8. The "transmission" $|a|^2$ (upper three curves) and $|b|^2$ (lower three curves) evaluated at $t = 0.4$ usec, for cases 1, 2, and 3.

acceptable. For hydrogen or tritium atoms, where the line shape is of little consequence, a much larger difference is acceptable.

Figure 6 shows the modulus and argument of $B_3$ as a function of $B - B_o$ or $\delta/2\pi$, for the same three cases. Note that the slopes of the $|A_3|$ and $|B_3|$ curves are opposite in sign for a given $\omega_3$.

Figure 7 shows the real and imaginary parts of the small decay constant $\mu_3$, as a function of $B - B_o$ or $\delta/2\pi$, for the same cases. Figure 8 shows the transmission $|a|^2$ and $|b|^2$ after a time $t = 0.4$ usec (12 cm). At a time as large as this, only the $\mu_3$ terms survive, so

$$|a|^2 = |A_3 e^{-\mu_3 t}|^2$$

and

$$|b|^2 = |B_3 e^{-\mu_3 t}|^2.$$
We now write simpler expressions for the special case of resonance ($\xi = 0$). As noted above, the coefficients do not vary rapidly, so some statements about the general nature of the solutions at resonance will apply approximately to the off-resonance solutions. We neglect the small matrix element $M$. For this case, the cubic equation becomes

$$u^3 - (\xi/2 - i\omega_b)e u^2 + f^2 u = 0,$$

where $f^2 = k_{RR}* + V*V*$ as before. The roots are

$$u_{1,2} = \left(\xi/4 - i\omega_b/e\right) \pm \sqrt{(\xi/4 - i\omega_b/e)^2 - f^2},$$

$$u_3 = 0,$$

which are seen to be consistent with the solutions obtained above for $\omega_b = 0$. Inserting these values for the $u_\xi$ in the general relations, we obtain

$$c_k = \frac{i_k R V*}{D_k}$$

and

$$\delta_k = \frac{-k_i R u_k}{D_k},$$

where

$$D_k = \frac{u_k (u_k - \omega_b e - i\xi/2) - V*V}{\frac{1}{2} (\xi + n)}$$

and $u_k$ have been defined as follows:

$$\frac{1}{2} (\xi + n)$$

$$\frac{1}{2} (\xi - n)$$

$$-V*V$$

$$-R/2V$$

The determinant $D$ may be written

$$D = \frac{\delta_i V* \xi}{R*V*} \left(1 + \frac{R^{*R}}{V*V*}\right).$$

For the initial condition $a = 1, b = 0, c = 0$, we obtain the coefficients

\[\begin{array}{lcc}
\hline
k & u_k & D_k \\
\hline
1 & \xi + n & R*R/4 \\
2 & \xi - n & R*R/4 \\
3 & 0 & -V*V \\
\hline
\end{array}\]
For large times, the solutions are therefore

\[ a \approx \frac{\sqrt{\nu}}{p^2} e^{-\nu_3 t} \]

\[ b \approx \frac{-\sqrt{\nu}}{2p^2} e^{-(\nu_3 + i\delta)t} \]

where \( \nu_3 \) is a rapidly varying function of \( \delta \) (see Figure 7).

For the initial conditions \( a = 0, b = 1, c = 0 \), we obtain the solutions

\[
\begin{array}{ccc}
1 & \frac{R^*R}{8p^2} (1 - \frac{\delta}{n}) & \frac{RV}{4p^2} (1 - \frac{\delta}{n}) & \frac{iR}{4n} \\
2 & \frac{R^*R}{8p^2} (1 + \frac{\delta}{n}) & \frac{RV}{4p^2} (1 + \frac{\delta}{n}) & -\frac{iR}{4n} \\
3 & \frac{RV}{p^2} & -\frac{RV}{2p^2} & 0 \\
\end{array}
\]

For the initial conditions \( a = 1, b = 0, c = 0 \), we obtain the solutions

\[
\begin{array}{ccc}
1 & \frac{R^*R}{4p^2} (1 - \frac{\delta}{n}) & \frac{RV}{4p^2} (1 - \frac{\delta}{n}) & \frac{iV}{2n} \\
2 & \frac{R^*R}{4p^2} (1 + \frac{\delta}{n}) & \frac{RV}{4p^2} (1 + \frac{\delta}{n}) & -\frac{iV}{2n} \\
3 & \frac{-R^*V}{2p^2} & \frac{R^*R}{4p^2} & \\
\end{array}
\]

From the symmetry of the equations, one can see that the relation between the sets of coefficients for the two assumed initial conditions must involve only the simultaneous interchange of \( R/2 \) with \( V \) and the definition of \( A_k \) with that of \( B_k \).

In general, the roots of the cubic equation are complex. The imaginary components correspond to (time dependent) energy shifts from the unperturbed eigenenergies characterizing the wave functions given in Table VIII. Consider, for illustrative purposes, the situation at crossing and resonance where

\[ \mu_{1,2} = \gamma/4 \pm \sqrt{(\gamma/4)^2 - p^2}, \quad \mu_3 = 0. \]

It is clear that if \( p^2 < (\gamma/4)^2 \), all the roots are real and the unperturbed energies remain correct. If, however, \( p^2 > (\gamma/4)^2 \), \( \mu_1 \) and \( \mu_2 \) will have an imaginary component and the level energies will be shifted. This is related to the phenomena of level repulsion in which, under certain circumstances, the energies of two states as a function of magnetic field do not cross, but rather repel, and thus interchange roles. This effect does not result in a shift of the position of the three-level resonance, however, since this is determined solely by the frequency at which \( \text{Re}(\mu_3) = 0 \), and this frequency will correspond* to \( \delta = 0 \) for any value of \( \omega_{\text{be}} \) or \( p^2 \). For \( \delta \neq 0 \), \( \mu_3 \) has an imaginary component (see Figure 7) and the slowly decaying states \( a \) and \( \delta \) may be regarded as slightly energy-shifted. The energy shift of the rapidly decaying components has no significance at large times.

Figures 9-11 illustrate the time dependence of \(|a|^2\) and \(|b|^2\) for the cases 1, 2, and 3 defined above (initial conditions \( a = 1, b = 0, \) and \( c = 0 \)). Figure 9 corresponds to resonance (\( \delta = 0 \)), Figure 10 corresponds to 1 G off resonance (\( \delta/2\pi = 2.8 \text{ MHz} \)), and Figure 11 corresponds to 9.6 G off resonance (\( \delta/2\pi = 26.9 \text{ MHz} \)). The last value is chosen for presentation since 9.6 G is approximately the difference in magnetic field values at which the various deuterium magnetic substates resonate.

Figure 12 illustrates the time dependence of \(|a|^2\) and \(|b|^2\) at resonance (\( \delta = 0 \)) for the cases 1, 2, and 3 but for the initial conditions \( a = 0, b = 1, \) and \( c = 0 \).

Figures 13-15 show the transmission of hydrogen metastable atoms versus magnetic field for an rf field of fixed frequency and strength (here taken to be 1610 MHz and 18.41 V/cm, respectively) and for several values of the transverse field (8.79, 8.99, 9.19, 9.39). As has been noted above, the apparent resonant frequency sometimes differs slightly from \( \delta = 0 \). This is due only to the slow variation of the coefficients \(|A_3|\) and \(|B_3|\) with frequency, and is unrelated to the energy shifts presently under discussion.
Fig. 9. The variation of $|a|^2$ and $|b|^2$ vs time for $B = B_0$ ($\delta = 0$) with initial conditions $a(0) = 1$, $b(0) = c(0) = 0$ for cases 1, 2, and 3.

Fig. 10. The variation of $|a|^2$ and $|b|^2$ vs time for $B - B_0 = -1$ G ($\delta/2\pi = 2.8$ MHz) with initial conditions $a(0) = 1$, $b(0) = c(0) = 0$ for cases 1, 2, and 3.

Fig. 11. The variation of $|a|^2$ and $|b|^2$ vs time for $B - B_0 = -9.6$ G ($\delta/2\pi \approx 26.9$ MHz) with initial conditions $a(0) = 1$, $b(0) = c(0) = 0$ for cases 1, 2, and 3.

Fig. 12. The variation of $|a|^2$ and $|b|^2$ vs time for $B - B_0$ ($\delta = 0$) with initial conditions $b(0) = 1$, $a(0) = c(0) = 0$ for cases 1, 2, and 3.
The transmission of hydrogen metastable atoms \(|a|^2 + |b|^2\) vs magnetic field for 1610 MHz; \(|R| = 250\) MHz, and \(|V| = 125\) MHz. The solid curves correspond to initial condition \(a(0) = 1\) and the dashed curves correspond to \(b(0) = 1\).

The peaks at different magnetic field strengths correspond, of course, to different nuclear spin substates. An interaction time of 0.4 usec (corresponding to a cavity length 12 cm) is assumed. The solid curves correspond to an initial pure alpha-state beam \((a(0) = 1)\) and the dashed curves correspond to a pure beta-state beam \((b(0) = 1)\). For the highly symmetric case \((|b|R| = |V|)\) shown in Figure 13, the two initial conditions result, except in the "tail" region, in identical solutions. The quantities \(|a|^2 + |b|^2\) and \(|a|^2\) are plotted in each case.

Figures 16-18 show the transmission of deuterium metastable atoms for the same cases and conditions.

Several observations about the general nature of the solutions may be made from the graphs:

1. For fixed \(|R|\), both the height and width of the lines which correspond to \(a(0) = 1\) increase with increasing \(|V|\). (For fixed \(|V|\), the height and width of the peaks which correspond to \(b(0) = 1\) increase with increasing \(|R|\), although this is not shown here.) The heights, of course, vary in the manner stated previously, and depend only on \(|R|/|V|\). For the case \(|b|R| = |V|\), the \(a(0) = 1\) and \(b(0) = 1\) solutions become nearly identical. This result is apparent from the symmetry of the equations.

2. From Figures 9-11, one can see that the width of the resonance lines must decrease monotonically as the interaction time in-
Fig. 16. The yield of deuterium metastable atoms \((|a|^2 + |b|^2\) and \(|a|^2\)) vs magnetic field for 1610 MHz; \(|R| = 250\) MHz, and \(|V| = 125\) MHz. For this case, curves corresponding to initial condition \(a(0) = 1\) and to \(b(0) = 1\) are identical.

Fig. 17. The yield of deuterium metastable atoms \((|a|^2 + |b|^2\) and \(|a|^2\)) vs magnetic field for 1610 MHz; \(|R| = 250\) MHz, and \(|V| = 250\) MHz. The solid curves correspond to initial condition \(a(0) = 1\) and the dashed curves to \(b(0) = 1\).

Fig. 18. The yield of deuterium metastable atoms \((|a|^2 + |b|^2\) and \(|a|^2\)) vs magnetic field for 1610 MHz; \(|R| = 250\) MHz, and \(|V| = 375\) MHz. The solid curves correspond to initial condition \(a(0) = 1\) and the dashed curves to \(b(0) = 1\).

The separation of metastable hydrogen or tritium atoms with different nuclear spin orientations appears to be very easy in the sense that the parameters may vary over a wide range. However, for metastable deuterium atoms, if one uses too large a field strength, the width of the lines will be too large. Thus, the minimum cavity length appears to be \(\sim 6\) cm for deuterium atoms, but could be shorter for hydrogen or tritium atoms. (This is because \(|R|/|V|\) must be held constant to achieve a given transmission at resonance. Since \(|V| < |V|_{\text{max}}\) is required, \(|R| < |R|_{\text{max}}\) is also required. But the decay constants corresponding to the unwanted nuclear spin states are approximately proportional to \(|R|\); this implies \(t > t_{\text{min}}\) where \(t\) is the cavity length.)

For an incident unpolarized beam of metastables (i.e., 1/2 of beam in the \(a\) state, 1/2 in the \(b\) state) the transmission at resonance of the "spin filter" is exactly 50\%, as may be verified from Figures 9 and 12 and from Figures 13-18. This follows from the expressions already derived which are repeated here in a matrix form:

3. The separation of metastable hydrogen or
\[
\begin{pmatrix}
    a(t+) \\
    b(t+)
\end{pmatrix} = \frac{1}{p^2} \begin{pmatrix}
    \nu y & -\nu y \\
    -\nu y & \nu y
\end{pmatrix} \begin{pmatrix}
    a(0) \\
    b(0)
\end{pmatrix}.
\]

The initial beam may be regarded as an incoherent mixture of \(\alpha\) and \(\beta\) states although the final beam is a coherent mixture. If we average over initial and sum over final states, we find

\[
|a(t+)|^2 + |b(t+)|^2 = \frac{1}{2} |a(0)|^2 + \frac{1}{2} |b(0)|^2;
\]

i.e., 50\% of an (electron) unpolarized beam is quenched.

We next consider the physical nature of the states which are transmitted through the spin filter. As a first step we eliminate dependence by defining the new variables

\[
\begin{align*}
A &= ae^{-i\omega t} \\
B &= be^{-i\omega_B t} \\
C &= ce^{-i\omega_C t}
\end{align*}
\]

The equations for these new variables are found to be

\[
\begin{align*}
iA &= (\omega - \omega)A + \nu R* C \\
iB &= \omega_B B + V*C \\
iC &= \nu RA + VB + (\omega e - \nu i\gamma)C
\end{align*}
\]

We choose \(\omega_B = 0\) (which we may do since the energy scale is arbitrary) and define as usual \(\delta = \omega_B - \omega\). The equations and definitions are then

\[
\begin{align*}
iA &= \delta A + \nu R* C \\
iB &= V*C \\
iC &= \nu RA + VB + (\omega e - \nu i\gamma)C
\end{align*}
\]

Let us define the spinors \(r\) and \(s\), with amplitudes \(p\) and \(q\), as follows:

\[
\begin{align*}
r &= \frac{1}{p} (\nu R*a' + V*s) \\
s &= \frac{1}{p} (V*a' - \nu R*s)
\end{align*}
\]

where the spinor \(a' = ae^{i\delta t}\). The total wave function, neglecting the \(e\) state (i.e., for \(t = \infty\)) is then of the form \(\psi = aa + bB = (ae^{-i\delta t} + bB = A(\nu e^{i\delta t}) + B\beta = Aa' + B\beta\). Notice that \(r\) and \(s\) are orthonormal. We can invert the definitions of \(r\) and \(s\) to find:

\[
\begin{align*}
a' &= \frac{1}{p} (\nu R*r + V*s) \\
b &= \frac{1}{p} (V*r - \nu R*s)
\end{align*}
\]

Substituting these expressions into the definition of \(\psi\):

\[
\psi = pr + qe = Aa' + B\beta
\]

we obtain:

\[
\begin{align*}
p &= \frac{1}{p} (\nu R* + V) \\
q &= \frac{1}{p} (V*A - \nu R*B)
\end{align*}
\]

We now derive the differential equations for \(p\) and \(q\):

\[
\begin{align*}
i \dot{p} &= PC + \frac{\nu S^A}{2p} \\
i \dot{q} &= \frac{V*S^A}{p} \\
i \dot{C} &= Pp - (\omega e + \nu i\gamma)C
\end{align*}
\]

If we specialize to resonance (\(\delta = 0\)) these equations become

\[
\begin{align*}
i \dot{p} &= PC \\
i \dot{q} &= 0 \\
i \dot{C} &= Pp - (\omega e + \nu i\gamma)C
\end{align*}
\]

The variables may be further separated as follows:

\[
i \dot{p} = P(i \dot{C}) = P(Pp - i(\omega e + \nu i\gamma)C) \\
= p^2 p + (\nu Y - i\omega e)p
\]
Thus our three equations are

\[ \dot{p} + (i\gamma - i\omega_{\beta e})p + p^2 p = 0 \]

Thus, the amplitude of the spinor \( s \) is conserved while the amplitude of \( r \) decays exactly as does the amplitude of \( e \).

Let us momentarily allow an arbitrary initial phase; i.e., we put, once again, \( R = R_0 \). In spinor notation, for \( \delta = 0 \), \( r \) and \( s \) may be written

\[ r = \frac{R_0}{\omega_e} e^{-i\omega_{\beta e} t} e^{i\delta t} \]

\[ s = \frac{R_0}{\omega_e} e^{-i\omega_{\beta e} t} e^{i\delta t} \]

The expectation values of the Pauli operators for these states are tabulated below:

<table>
<thead>
<tr>
<th>( \langle \sigma_x \rangle )</th>
<th>( r )</th>
<th>( s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon_{10} \cos(\omega_{\beta e} t + \delta) )</td>
<td>( \epsilon_{10} \cos(\omega_{\beta e} t + \delta) )</td>
<td>( -\epsilon_{10} \cos(\omega_{\beta e} t + \delta) )</td>
</tr>
<tr>
<td>( \langle \sigma_y \rangle )</td>
<td>( \epsilon_{i1} \sin(\omega_{\beta e} t + \delta) )</td>
<td>( -\epsilon_{i1} \sin(\omega_{\beta e} t + \delta) )</td>
</tr>
<tr>
<td>( \langle \sigma_z \rangle )</td>
<td>( \epsilon_{i1} )</td>
<td>( -\epsilon_{i1} )</td>
</tr>
</tbody>
</table>
are of roughly equal importance. Thus the simplest equations which reasonably describe the four level system are:

\[ i(x_+ \delta t) + \omega_+ t \]
\[ \dot{i} = \frac{1}{2} \dot{y}_{ae} e - \frac{1}{2} \dot{y}_{be} \cos(\omega_{ab} \delta t) \]
\[ \dot{i} = \frac{1}{2} \dot{y}_{ae} e + \frac{1}{2} \dot{y}_{be} \cos(\omega_{ab} \delta t) \]

where we have put \( R' = -R \) and \( V' = V \) in accordance with Table X and the discussion in Section 6. Note that these relations hold only in the zero field limit; i.e., if \( \epsilon_1, \epsilon_2, \epsilon_3, \) and \( \epsilon_4 + 1 \). All four-level calculations presented here are based on this assumption. Note also that an arbitrary initial phase cannot be included in the four-level case by putting \( R = R_0 \) as was done in Section 7. The initial phase is, however, of no importance.

We first consider the quenching of a states by a transverse electric field with no rf longitudinal field present (i.e., \( R = 0 \)). The equations above then separate into two independent pairs of which the \( a - f \) pair is

\[ \dot{i} = \dot{y}_{af} e \]
\[ \dot{i} = \dot{y}_{ac} e \]

These equations are easily solved if one assumes

\[ a = A_1 e + A_2 e^{-\mu_2 t} \]
\[ d = (D_1 e + D_2 e^{-\mu_2 t}) e^{-\mu_2 t} \]

One finds that \( \mu_1 \) and \( \mu_2 \) are the roots of

\[ \mu^2 + 2\mu + |V|^2 = 0 \]

where

\[ \xi = \frac{1}{2} \omega_{af} \frac{1}{2} \]

Thus we may write \( \mu_1, \mu_2 = \xi \pm \eta \) where \( \eta = \sqrt{\xi^2 - |V|^2} \).

For the initial condition \( a(0) = 1, d(0) = 0 \), the general solution for \( a \) is found to be

\[ a = a(1 - \frac{5}{6}) e^{-(\xi + \eta)t} + b(1 + \frac{5}{6}) e^{-(\xi - \eta)t} \]

If \( |\xi|^2 \gg |V|^2 \), we may approximate \( a \) by

\[ a = \xi - \frac{|V|^2}{2\xi} \]

The expression for \( a \) becomes

\[ a \approx \frac{|V|^2}{4\xi^2} e^{2\xi t} + e^{-\frac{|V|^2}{2\xi}t} \]

The first term has a small coefficient and rapid decay constant; for times of interest, we may write

\[ |a|^2 \approx e^{-(\xi^2) t} \]

which is the Stark quenching formula given by Lamb and Rutherford. Similar results are obtained for the \( b - e \) quenching:

\[ |b|^2 \approx e^{-(\xi^2) t} \]

and for the \( a - e \) quenching:

\[ |a|^2 \approx e^{-(\xi^2) t} \]

This result is not applicable to the \( b - f \) quenching, however. It should also be noted that for the field strength of present interest, the expansion of \( \eta \) used above will not be valid for the \( b - e \) case if \( \omega_{be} \) is small or zero.

Thus, for a atoms in the presence of both rf and static electric fields, we would expect an effective decay constant of the order of, but greater than, \( \gamma_{af} \). The situation is complicated, however, since the various contributions to the decay are coherent.

An analytic solution at \( \delta = 0 \) is possible if \( \frac{1}{2} \dot{y}_{be} e \cos(\omega_{ab} \delta t) \) is approximated by \( e^{-\mu_2 t} \). As already noted, this is not a good approximation; however, it at least partially takes into account the \( b - f \) interaction and is included here primarily for the physical insight that it may afford.

To obtain this solution, we first eliminate the oscillating time dependence with the substitution
The equations for these variables are

\[
\begin{bmatrix}
  iA \\
  iB \\
  iC \\
  iD
\end{bmatrix} =
\begin{bmatrix}
  \delta & 0 & \Re & \gamma \\
  0 & 0 & \Re & -\Re \\
  -\Im & \Im & 0 & 0 \\
  \Im & -\Im & \Re & -\Re + \Im \gamma
\end{bmatrix}
\begin{bmatrix}
  A \\
  B \\
  C \\
  D
\end{bmatrix}
\]

In terms of the previously defined amplitudes \( p \) and \( q \), we can derive

\[
i\dot{p} = R6A/2 + PC
\]
\[
i\dot{q} = V*6A/P + PD
\]
\[
i\dot{c} = Fp - (\omega_B + \Im \gamma)C
\]
\[
i\dot{d} = Fq - (\omega_B - \delta + \Im \gamma)D
\]

At resonance (\( \delta = 0 \)) the equations reduce to the two coupled pairs

\[
i\dot{p} = PC
\]
\[
i\dot{c} = Fp - (\omega_B + \Im \gamma)C
\]
and

\[
i\dot{d} = PD
\]
\[
i\dot{d} = Fq - (\omega_B + \Im \gamma)C
\]

from which it follows that

\[
\begin{align*}
  \dot{p} + 2\dot{\omega}_B p + \omega_B^2 p &= 0 \\
  \dot{c} + 2\dot{\omega}_B c + \omega_B^2 c &= 0
\end{align*}
\]

and

\[
\begin{align*}
  q + 2\dot{\omega}_B q + \omega_B^2 q &= 0 \\
  \ddot{d} + 2\dot{\omega}_B d + \omega_B^2 d &= 0
\end{align*}
\]

Thus

\[
p = p_1 e^{-\mu_1 t} + p_2 e^{-\mu_2 t}
\]

where \( \mu_1,2 = \xi_\text{ef} \pm \sqrt{\xi_\text{ef}^2 - \omega_B^2} \)

and

\[
q = q_1 e^{-\mu_1 t} + q_2 e^{-\mu_2 t}
\]

where \( \mu_1,2 = \xi_\text{ef} \pm \sqrt{\xi_\text{ef}^2 - \omega_B^2} \)

The quantities \( \mu_1 \) and \( \mu_2 \) are recognized as the two larger decay constants discussed in the three-level case; thus, for times of interest, \( p > 0 \). However, \( \xi_\text{ef} >> \omega_B \) for the present region of interest, so we may expand the square root (as before) to obtain

\[
\begin{align*}
  \mu_1 &\approx 2\xi = \gamma/2 - \Im \omega_B \\
  \mu_2 &\approx \omega_B^2/2\xi = \omega_B^2/(\gamma/2 - \Im \omega_B)
\end{align*}
\]

The \( \mu_1 \) term decays rapidly, so for times of interest

\[
q \approx q_2 e^{-(\omega_B^2/2\xi) t}
\]

Our initial conditions are \( A(0) = 1, B(0) = C(0) = D(0) = 0 \). Now

\[
q(0) = \frac{1}{F} (V^*A(0) - \Re R^*B(0)) = \frac{V^*}{F}
\]

and

\[
q(0) = \frac{1}{F} (V^*A(0) - \Re R^*B(0)) = 0
\]

where the latter condition follows from the differential equations. These conditions yield

\[
q_2 = \frac{r + n}{2n} \frac{V^*}{F} \approx \frac{V^*}{F}
\]

Thus, for large times,

\[
p \to 0; \quad q \to \frac{V^*}{F} e^{-(\omega_B^2/2\xi) t}
\]
Fig. 19. The line shape as calculated with various approximations. The upper curves correspond to $|a|^2$ and the lower curves to $|b|^2$. The curves marked 1 correspond to the three-level theory. The curves marked 2 show the effect of inclusion of both terms of $\cos \omega t$ in the three-level theory. The curves marked 3 correspond to the approximate four-level theory in which $e^{i\omega ft}$ $\cos \omega t$ is

$$e^{i(\omega ft - \delta)t}$$

replaced by $e^{i\omega ft}$ and $e^{i(\omega ft + \delta)t}$. The curves marked 4 correspond to the exact solutions of the four-level problem.
Inserting these into the definition of $A$ and $B$, we obtain

$$A = \frac{1}{\hbar} \left( \frac{\omega R}{2} P + Vq \right) + \frac{|V|^2}{\hbar^2} e^{-(P^2/2\hbar)t}$$

$$B = \frac{1}{\hbar} \left( V^* P - \omega Rq \right) + \frac{RV}{2P} e^{-(P^2/2\hbar)t}.$$ 

In the three-level case, for large time, we obtained

$$A = \frac{|V|^2}{\hbar^2} B = \frac{-RV}{2P}.$$ 

Thus, both $|A|^2$ and $|B|^2$ are altered only by a multiplicative factor $e^{-\gamma_{af}t}$ where

$$\gamma_{af} = \frac{\hbar^2}{\omega_{af} + (\gamma/2)^2}, \quad P^2 = kR^2 + |V|^2.$$ 

A point worth noticing with respect to these solutions is that only the state with amplitude $q$ (the $s$ state) is coupled directly to the $f$ state. That is, if the $e$ level were not present, the $a$-$f$-$\beta$ interaction, in this approximation, would select the $r$ state and quench the $s$ state. This is exactly the opposite to the situation for the $a$-$e$-$\beta$ interactions; thus this contribution from the $f$ level is destructive.

In Figure 19 the line shape obtained by numerical integration, with various approximations, is shown. The calculation is for the case of $n_L = 0$ deuterium atoms at the (nearly) optimum frequency of 1611.99 MHz (which corresponds to $B_0 = 575$ G). The parameters are $|R| = |V| = 250$ MHz. We note that, as expected, the curve which corresponds to the ($\delta = 0$) analytic solution given above gives a result about midway between the three-level and the four-level results.

In Figure 20 the exact four-level results for the loss (at resonance) of $|a|^2$ are shown for various parameters and for the same frequency and magnetic field as above. These curves represent the ratio of $|a|^2$ (at 0.25 usec) to the three-level equilibrium value ($|a_o|^2$) which would obtain in the absence of the $f$ level. The dashed curve represents the prediction of the approximate four-level analytic solution given above. Note that this predicted value depends only on $k|R|^2 + |V|^2$. The ratios vary approximately linearly with the interaction time $t$. The ratio of $|b|^2$ to $|b_o|^2$ is, for this case, indistinguishable from the ratio of $|a|^2$ to $|a_o|^2$.

Figure 21 shows the time dependence of the solutions, at $\delta = 0$, for the conditions used in Figure 19. Note that the inclusion of the antiresonant term in the three-level theory (the Bloch-Siegert correction term) results in a decay from the three-level equilibrium solution of about 2%/usec.

Figure 22 shows the line shape for a frequency of 1508.326 MHz. Again the calculation is for $m_L = 0$ deuterons (for which this particular frequency corresponds to $B_0 = 536$ G) and for $|R| = |V| = 250$ MHz. Note the shift of the peaks from the resonant field.

Figure 23 shows the loss through the $f$ state, for a frequency 1508.326 MHz and field 538 G ($m_L = 0$ deuterium atoms), for a variety of parameters $|R|$ and $|V|$. In this case, unlike that shown in Figure 20, the ratios $|a|^2/|a_o|^2$ and $|b|^2/|b_o|^2$ are not identical. Further, the loss through the $f$ level appears to be somewhat greater. However, as may be seen by comparing Figures 19 and 22, the peak positions are shifted in the latter case, and therefore $\delta = 0$ does not, in general, correspond to maximum transmission. If both $|a|^2$ and $|b|^2$ transmission
Fig. 21. The time dependence of the amplitudes $|a|^2$ and $|b|^2$, at $\delta = 0$, for various approximations. The curves are labeled as in Fig. 19.
Fig. 22. The line shape as calculated with various approximations. The curves are labeled as in Fig. 19. Note the shift of the peaks from the resonant field (see arrows).
1.0
~ 0.9~
~
'i'
El
EQUILIBRIUM 3 LEVEL THEORY
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0

Fig. 23. The lower graph shows the ratio of $|a|^2$ to $|a_o|^2$, where $|a_o|^2$ is the three-level equilibrium value of $|a|^2$, for $B_0 = 538 \, G$. The upper graph shows the ratio of $|b|^2$ to $|b_o|^2$ for these conditions.

are of interest (see Section 9) the value of $\delta$ which results in maximum transmission of $|a|^2 + |b|^2$ will depend on the parameters. (This is true since the $|a|^2$ and $|b|^2$ curves are shifted in opposite directions for $\omega_{ab} \neq 0$, and the relative contribution of $|a|^2$ and $|b|^2$ to the total transmitted beam depends on $|R|/|V|$.) Thus the curves given in Figure 23 predict greater loss, in general, than would be obtained by choosing an optimum value of $\delta$.

Finally, calculations show that the three-level and four-level results agree to within $\pm 10\%$, for a wide range of parameters, in the tail region. Thus, the transmission of the unselected substates is adequately described by the results given in Table XI. The transmission of the selected substate can, however, be improved by the field shaping technique to be described in Section 9.

9. ADIABATIC VARIATION OF THE ELECTRIC FIELDS

In the preceding discussion, we have assumed that the various applied fields are constant throughout the spin filter. If this condition is not met, we must resort to numerical techniques to solve even the three-level equations, although some general features of the solutions may be deduced from the form of the equations.

For application to a practical polarized ion source, the optimum transmission of the desired nuclear spin substate can be achieved if (a) the static electric field is constant, and (b) the rf field increases slowly from zero at the entrance of the spin filter to a maximum near the center and then decreases to zero at the exit.

It was shown in Section 7 that, in three-level approximation, exactly 50% of an (electron) unpolarized $^2$H beam with the desired $m\_1$ value could be transmitted through a combination of static transverse and longitudinal rf electric fields. In a practical ion source, however, the $\beta$ component of the atomic beam will almost certainly be quenched by the required "sweep" fields long before it reaches the spin filter. In addition, any $\beta$ component which emerges from the filter will probably be quenched before reaching the argon exchange cell.

For the parameters $|R| = |V| = 250 \, MHz$, for example, only 64% of an initially pure $a$ beam would emerge from the spin filter in an $a$ state, so that only about 1/3 of the initially produced unpolarized beam would be available at the argon exchange cell. (Note that we are referring always to the beam component with the desired $m\_1$ value; thus, in terms of the total atomic beam the 1/3 given above becomes 1/6 for hydrogen or tritium beams and 1/9 for deuterium beams.)

If the fields are shaped as indicated above, it is possible, in the three-level approximation, to achieve 100% transmission for a pure $a$ beam. A spin filter with such field shaping will have 0% transmission for a $\beta$ beam and thus will still have 50% transmission for an (electron) unpolarized beam. This is indicated in Figure 24 for a space variation of the rf field strength of the form $\sin(2\pi z)$ where $z$ is the distance from the entrance to the rf region and $z_0$ is the total length of the rf region. For this example $|V|$ is assumed to have the constant...
Fig. 24. The quantities $|a|^2 + |b|^2$ and $|a|^2$ vs time as a metastable beam ($a(0) = 1$) traverses a cavity whose rf strength varies as $\sin(\pi z/\tau_0)$, where $\tau_0 = 30$ cm is the total length of the cavity. These curves correspond to $\delta/2\pi = 0$, $|R|_{\text{max}} = 250$ MHz, and to a constant static electric field such that $|V| = 250$ MHz.

value 250 MHz and $|R|_{\text{max}} = 250$ MHz. The results of both three-level and four-level theory are shown. (The deviation from 100% of the transmission which corresponds to three-level theory arises solely from the inclusion of both frequency terms in the expansion of $\cos(\omega t)$, while the discussion above is based on the assumption $\cos(\omega t) \equiv \frac{1}{2} e^{-i\omega t}$. All results presented in Figures 24 and 25 are based on calculations which include both terms of $\cos(\omega t)$.)

These results can be understood as follows. For simplicity, consider the special case of resonance ($\delta = 0$) and crossing ($\omega_{de} = 0$). The three-level equations are then, as noted in Section 7:

1. $\dot{a} = \hbar R a c$
2. $\dot{b} = V b c$
3. $\dot{c} = \hbar R a + V b - \hbar (i c)$.

Also as noted in Section 7, it is evident that one possible solution of these equations is

$a = a_0$, $b = b_0$, $c = 0$

where the constants $a_0$ and $b_0$ are related by

$\hbar R a_0 + V b_0 = 0$

or, equivalently

$a_0/b_0 = -2V/R$.

(These are, in fact, the equilibrium solutions previously discussed.) If $|R| \rightarrow 0$, the equilibrium solution will correspond to a pure $a$ state. If $|R| \rightarrow 0$ all nuclear substates are equivalent and no selection would occur. However, if $|R|$ is increased sufficiently gradually, so that the condition $\hbar R a + V b = 0$ can be followed adiabatically, the nuclear spin selection can be made without loss. At this point, the amplitude which describes a particle in the beam will be a coherent mixture of $a$ and $b$ states. If $|R|$ is then slowly decreased to zero, the $a$ and $b$ mixture will be transformed back into a pure $a$ state. Thus there are two important aspects to the field shaping: (a) a slow rise of $|R|$ prevents loss from occurring through the excitation of "transients" as the beam enters the cavity,
and (b) a slow fall of $|R|$ transforms the mixed $a$ and $\beta$ state back into the more stable pure $a$ state. If no quenching between the rf region and the argon exchange region were likely to occur, the second part of the field shaping would be unnecessary. It is evident that the important parameter in these arguments is $|R|/|V|$, so that an exactly constant $|V|$ is not required.

The above can also be expressed in terms of the $r$ and $s$ spinors defined in Section 7. We recall that

$$s = \frac{1}{F} (Va - kRb)$$

and has the constant amplitude $q_1$. In terms of $a$ and $b$, for $\delta = 0$,

$$q = q_1 = \frac{1}{F} (V*a - kRb) .$$

Thus, if $|R| > 0$, $s \rightarrow a$ and $q(t) = q_o = a(0) = 1$. If $|R|$ is increased sufficiently gradually, the state $s$ will be conserved and therefore a particular $a + \beta$ mixture will be formed. If $|R|$ is then decreased sufficiently gradually, it will again be conserved and the mixture state will be transformed back into a pure $a$ state.

It remains to determine what is meant by "sufficiently gradually." In Figure 25 the results for various assumed rise times for the rf field are presented. The exact four-level theory was used for these calculations. We assume a 30 cm overall path (velocity = 30 cm/µsec) and that only the static electric field acts over the last 15 cm. The rising and falling portions of the rf field is assumed to have the shape $\sin^2(t\pi/2T)$. The rf field is fully turned off at $t = 0$ and at $t = 0.5$ µsec. The various curves are labeled with the parameter $T$ and in each case the upper curve represents $|a|^2 + |b|^2$, and the lower curve $|a|^2$. It is seen that no loss occurs for the case $T = 0.2$ µsec. At the steepest part of the $\sin^2(t\pi/2T)$ curve, the fractional rate of change in $|R|$ is $\pi/T$. For $T = 0.2$ µsec, $\pi/T = 15.7 \times 10^6$ sec$^{-1}$. Since the Larmor frequency is around $1600 \times 10^6$ sec$^{-1}$, $|R|$ changes about 1% in a Larmor cycle. Thus, we have demonstrated that a satisfactory criterion for adiabaticity is that the strength of the rf field may change no more than about 1% per cycle. This is about the value that one would expect.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge many helpful discussions with the following Los Alamos Scientific Laboratory colleagues: D. Dodder, B. Watt, R. Stevens, L. Heller, and J. Jackson. Professor C. Drake of Oregon State University also made some useful comments concerning this material. H. Butler kindly made available a very useful computer program for numerical integration.

REFERENCES

A FORTRAN IV program which evaluates the solutions developed in Section 7 is given in this appendix. The version given computes the squared amplitudes $|a|^2$, $|b|^2$, $|c|^2$, and $|a|^2 + |b|^2$ as a function of time for a set of fixed driving frequencies and for fixed magnetic field. The modulus and phase of $a$, $b$, and $c$ are also given. Other versions of the program exist in which time is held constant while the variation of the above quantities with magnetic field at fixed frequency, or with frequency at fixed magnetic field, is studied.

The input for the program is as follows:

Card 1 (FORMAT 6 F12.6)

XSPIN spin of nucleus (i.e., $\frac{1}{2}$ or 1)

GJ $g_J$ value for the $2S_{\frac{1}{2}}$ states ($\frac{\pi}{2}$)

XMU magnetic moment of the nucleus (in nuclear magnetons)

DELW zero magnetic field hyperfine splitting for the $2S_{\frac{1}{2}}$ state (MHz)

GJP $g_J$ value for the $2P_{\frac{3}{2}}$ states (2/3)

DELWP zero magnetic field hyperfine splitting for the $2P_{\frac{3}{2}}$ states (MHz)

(The radiative correction to the $g_J$ value is supplied by the subroutine BREIT and should not be included in GJ and GJP.)

Card 2 (FORMAT 6 F12.6)

FREQMN minimum applied frequency (MHz)

FREQDL increment in applied frequency (MHz)

FREQMX maximum applied frequency (MHz)

TF maximum time at which solutions are to be evaluated (psec)

DELT increment in time at which solutions are to be evaluated (TF/DELT should not exceed 500)

BGAUSS magnetic field (G)

Card 3 (FORMAT 6 F12.6)

XM $M$ in MHz (angular frequency)

RR $R$ in MHz (angular frequency)

VV $V$ in MHz (angular frequency)

where these quantities are complex and therefore appear as three pairs of numbers on the card. The real part of each quantity appears first. The relations between these units and practical units are given in the program listing.

Card 4 (FORMAT 6 Ib)

IMODE If IMODE = 1, program returns to start.

If IMODE = 2, program returns to read in new Card 3 and proceeds.

ICSMN minimum case to be calculated

ICSMX maximum case to be calculated

where $m_i = 1, 0, -1$ correspond to cases 1, 2, and 3 for deuterium atoms and $m_i = \frac{1}{2}, -\frac{1}{2}$, correspond to cases 1 and 2 for hydrogen and for tritium atoms.

The program consists of a main program together with several subroutines. The function of the various subroutines is as follows:

a) SUBROUTINE CUBIC (P, Q, R, RT1, RT2, RT3)

This subroutine evaluates the solutions of a cubic equation with complex coefficients of the form $x^3 + Px^2 + Qx + R = 0$. It uses double-precision arithmetic in order to obtain the required accuracy. The three complex roots, RT1, RT2, and RT3 are in order of decreasing real parts.

b) SUBROUTINE DPROD (XR, XI, YR, YI, ZR, ZI)

This subroutine multiplies the complex numbers X and Y together to give complex Z. Double-precision arithmetic is used; thus the real and imaginary parts are carried separately.

c) SUBROUTINE DARCTAN (Y, X, Z)

This subroutine finds $Z = \arctan (Y/X)$ in the correct quadrant. Double-precision arithmetic is used.

d) SUBROUTINE BREIT (XI, FFF, XM, GJ, GI, DELW BGAUSS, W, XGAUSS, EPS1)

This subroutine evaluates the energy of a given state according to the Breit-Rabi formula (see Section 2). The input variables are

XI spin of nucleus

FFF F quantum number

XM $m_F$ quantum number

GJ $g_J$ value (atomic g factor) excluding radiative corrections

GI $g_I$ value (nuclear g factor)

DELW zero magnetic field hyperfine splitting in MHz

BGAUSS magnetic field in G

The output variables are

W energy of state in MHz

XGAUSS value of the parameter X (defined in Section 2)
The EPS1 nuclear moment correction term $c_1$ (in MHz; already included in W) is given. The deviation of these quantities from zero gives some idea of the accuracy with which the roots have been determined.

Tape 10 is defined as input and Tape 9 as output for the particular system for which this program was written.

```
C TIME PLOT VERSION
C DIVIDE R AND V BY 13.94 TO OBTAIN VOLTS/CM PEAK TO ZERO
C DIVIDE M BY 8.80 TO OBTAIN GAUSS
C "HAT IS, ENTER R,V, AND M IN MHZ ANGULAR FREQUENCY
DIMENSION FFA(6),FFB(6),XMFA(6),XMFB(6)
DIMENSION FARS(3),FACS(3),FBSCS(3)
DIMENSION ASQ(500),BSQ(500),CSQ(500),ABSQ(500),TIME(500)
COMPLEX XMM,RRSVV,DELAC,P,G,R,XXMSTR,XXRSTAR,VVSTAR,XI,RT1,RT2,RT3,
1XINU(3),DEPS(3),UEL(3),A(3),B(3),C(3),SUM
COMPLEX X,F1,F2,F3,AA,AB,CC,PHASAH,PHASAC
DOUBLE PRECISION PI
COMMON PI
1 FORMAT(6F12.6)
2 FORMAT(6H A(K)= 6F9.4,7H Mu(l)= 2F9.4,4H F1= 2F9.4)
3 FORMAT(6H B(K)= 6F9.4,7H Mu(2)= 2F9.4,4H F2= 2F9.4)
4 FORMAT(6H C(K)= 6E'9.4,7H Mu(3)= 2F9.4,4H F3= 2F9.4)
5 FORMAT(6H OFRECJ= F10.3,5X,3HK=1, 15X,3HK=2, 15X,3HK=3 )
6 FORMAT(122H0 TIME A*+2 B.*2 C**? A**2 MOD A PHASE A MUII B PHASE B MOD C PHASE C
7) 1F12.3.4F12.6,6F10.3)
7 FORMAT(F12.3.4F12.6,6F10.3)
8 FORMAT(1H1)
9 FORMAT(4H MM= 2F12.6,4H RR= 2H12.6,4H VV= 2F12.6)
10 FORMAT(6H FREQ= F12.3,7H GAMMA= F12.3,5H FAB= F12.3, 5H FAC= F12.
13.5H FBC= F12.3 )
12 FORMAT(4H DOUBLE PRECISION CUBIC SOLUTION METHOD )
13 FORMAT(6I4)
14 FORMAT(7H SPIN= 12.6,4H GJ= F12.6,5H MU= F12.6,6H DELW= F12.6,
15H GJP= F12.6,7H DELWP= F12.6)
15 FORMAT(14H STATE NUMBER 14,9H AT FIELD F7.1, 6H GAUSS )
16I4 FORMAT(48H INITIAL CONDITIONS A=1, B=0, C=0)
16I4 FORMAT(48H INITIAL CONDITIONS A=0, B=1, C=0)
PI=4.0*DATAN(I.0D+0)
GAMMA=200.0*3.1415927
XI=CPXLX1.0.0,1.0)
WRITE(9,8)
19 RREAD(10.1)XSPIN,GJ,XMU,DELW,GJP,DELWP
READ(10.1)FREDMN,FREDQL,FREDMX,FF,DELT,GAUSS
G1=XMUXSPIN
ISPIN=xSPIN*1.0
GO TO (100,101),ISPIN
```

44
XMFB(1) = 0.5
FFA(2) = 1.5
XMFA(2) = 0.5
FFB(2) = 0.5
XMFB(2) = -0.5
FFA(3) = 1.5
XMFA(3) = -0.5
FFB(3) = 1.5
XMFB(3) = -1.5

READ(10,1) XMM, RR, VV
READ(10,13) IMODE, ICSMN, ICSMX

IF IMODE = 1 GOTO 19
20 GOTO 20

WRITE(9,6)

IF (IY) = 1 GO TO 20
NGFREG = FREQMN - FREQMN/FREQL
NGFREG = NGFREG + 1
NTIME = TF/DELT
NTIME = NTIME + 1

DO 132 INIT = 1, 2
DO 110 ICASE = ICSMN, ICSMX
CALL BREIT(XSPIN, FFA(ICASE), XMFA(ICASE), GJ, GI, DELW, BGAUSS, FA, XGAUS
15, EPS1)
CALL BREIT(XSPIN, FFB(ICASE), XMFB(ICASE), GJ, GI, DELW, BGAUSS, FB, XGAUS
15, EPS1)
CALL BREIT(XSPIN, FFA(ICASE), XMFA(ICASE), GJ, GI, DELWP, HGAUSS, FC, BGA
1USS, EPS1)

FC = FC - 1058.070
FAR = FFB - FFA
FAC = FAC - FC
FAB = (ICASE) * FAB
FACS(ICASE) = FAC
FBCS(ICASE) = FFB - FC

DO 110 I = 1, NGFREG
F1 = I - 1
FREQ = FREQM(F1) * FREQL

DIFLAB = FREQM(F1) * FREQL

P VAR = CMPLX(0.0, 0.0)
A1 = 0.5 * GAMMA

A1M = (DELAR * XMM + XMSTR)
A1M = CONJG(A1M)

RRSTAR = CONJG(RR)

VVARSTAR = CONJG(VV)

PDEL = SDEL + SDELA

Q = SDEL + SDELA - RR*VVARSTAR = 0.25*(RR*RRSTAR + XMM*XHMSR)

R = SDELA - RR*VVARSTAR*0.25*(RR*RRSTAR + XMM*XHMSR*DELAC)

P = X1*P
Q = 0
R = X1*R

CALL CUBIC(P, Q, R, RT1, RT2, RT3)

X == RT1
F1 = X**3 + P*X**2 + 0*X + R
X = RT2
F2 = X**3 + P*X**2 + 0*X + R
X = RT3
F3 = X**3 + P*X**2 + 0*X + R

XMU(1) = RT1
XMU(2) = RT2
XMU(3) = RT3

DO 22 K = 1, 3
D = (DELAC*XMU(K) + DELAR*XMU(K) + RR*VVSTAR)

EPS(K) = 0.5*(XMU(K)*DELAC - RR*VVARSTAR)/D
A1 = EPS(2)*DEL(3) - EPS(3)*DEL(2)
A1 = EPS(3)*DEL(1) - EPS(1)*DEL(3)
A1 = EPS(1)*DEL(2) - EPS(2)*DEL(1)
SUM = A1 + A2 + A3

GOTO (130, 131, INIT)

A1 = DEL(2) - DEL(3)
A2 = DEL(3) - DEL(1)
A3 = DEL(1) - DEL(2)

DO 23 K = 1, 3

45
A(K)=A(K)/SUM
B(K)=EPS(K)+A(K)
C(K)=DFL(K)*A(K)
WRITE(9,8)
WRITE(9,14)XSPIN, GJ, XMU, DELW, GJO, DELWP
WRITE(9,5)FREQ
WRITE(9,2) (A(K), K=1,3), RT1, F1
WRITE(9,3) (B(K), K=1,3), RT2, F2
WRITE(9,4) (C(K), K=1,3), RT3, F3
GO TO (150, 151), INIT
WRITE(9, 160)
GO TO 152
WRITE(9, 161)
WRITE(9, 9) XI MM, HR, VV
WRITE(9, 10) FRLU, GAMMA, FABS(ICASE), FACS(ICASE), FBCS(ICASE)
WRITE(9, 15) ICASE, NGAUSS
WRITE(9, 6)
DO 114 ITIME=1, NTIME
FITIME=ITIME
TIMF(ITIME)=T
X=RT1*T
F1=CEXP(X)
X=RT2*T
F2=CEXP(X)
X=RT3*T
F3=CEXP(X)
AA=A(1)/F1+A(2)/F2+A(3)/F3
BB=B(1)/F1+B(2)/F2+B(3)/F3
CC=C(1)/F1+C(2)/F2+C(3)/F3
CC=CC*CEXP(PHASAC*T)
ASQ(ITIME)=(CABS(AA))**2
BSQ(ITIME)=(CABS(BB))**2
CSQ(ITIME)=(CABS(CC))**2
ARSQ(ITIME)=ABS(ASQ(ITIME)+BSQ(ITIME)
XMODAA=CABS(AA)
XMODBB=CABS(BB)
XMODCC=CABS(CC)
PHASAA=ATAN2(AIMAG(AA), REAL(AA))
PHASAA=180.0*PHASAA/3.1415927
PHASBB=ATAN2(AIMAG(BB), REAL(BB))
PHASBB=180.0*PHASBB/3.1415927
PHASCC=ATAN2(AIMAG(CC), REAL(CC))
PHASCC=180.0*PHASCC/3.1415927
WRITE(9, 7) TIME(ITIME), ASQ(ITIME), BSQ(ITIME), CSQ(ITIME), ARSQ(ITIME)
CONTINUE
GO TO (19, 20), !MODE
END

SUBROUTINE CUBIC(P, O, R, RT1, RT2, RT3)
DOUBLE PRECISION VERSION
C SOLVES CUBIC EQUATIONS OF THE FORM X**3+P*X**2+Q*X+R=0.0 WITH P, Q, C
AND R COMPLEX, THE THREE ROOTS RT1, RT2, AND RT3 ARE IN ORDER OF C
INCREASING REAL PART.
COMPLEX P, O, R, RT1, RT2, RT3
DOUBLE PRECISION PR, PI, OR, QI, HR, R1, P2R-P2I, AR, AI, P3R, P3I, PQR, PPI,
1R, RI, R2R, R2I, R2I, RTABS, RTARG, RTR, RTI, AAR,
2AA1, DARG
W120=CMPLX(-0.5, 0.8660254)
PR=REAL(P)
QR=REAL(Q)
RR=REAL(R)
PI=AIMAG(P)
QI=AIMAG(Q)
RI=AIMAG(R)
CALL DPROD(PR, PI, PR, P2R, P2I)
AR=QR/3.0-P2R/9.0
AI=O1/3.0-P21/9.0
ASNL=AR*3.0
ASNL1=AI*3.0
A=CMPLX(ASNL,ASNL1)
CALL DPROD(P2R,P21,PR,PR1,P3R,P31)
CALL DPROD(PR,P1,UR,QR,POR,PQ1)
BR=P3R/27.0-PQR/6.0*RR/2.0
BI=P31/27.0-PQ1/6.0*RI/2.0
CALL DPROD(BR,B1,BR1,B2R,B21)
CALL DPROD(AI,AR,A1,AR1,A2R,A21)
CALL DPROD(AI,AR,A1,AR1,A3R,A31)
RT2R=B2R+A3R
RT21=B21+A31
RTABS=(RT2R**2+RT21**2)**0.25
CALL DARCNT(R121,RT2R,RTARG)
RTARG=RTARG/2.0
RTR=RTABS*DCOS(RTARG)
RTI=RTABS*DSIN(RTARG)
ABSAA=DSORT((RTR-BR)**2+(RTI-B1)**2)
ABSBB=DSORT((RTR+RR)**2+(RTI+RI)**2)
IF(ABSAA.GE.ABSBB)GO TO 2
SGN=-1.0
GO TO 3
2 SGN=1.0
3 AAR=RR+SGN*RTI
AAI=BI+SGN*RTR
ARS=(AAR**2+AAI**2)**(1.0/6.0)
CALL DARCNT(AAI,AAR,ARG)
ARG=ARG/3.0
UR=ARS*DCOS(ARG)
UI=ARS*DSIN(ARG)
U=CMPLX(UR,UI)
DO 1 J=1,3
RT(J)=U-(P+U/A)/3.0
1 U=U*W120
R1=REAL(RT(1))
R2=REAL(RT(2))
R3=REAL(RT(3))
IF(R1.GE.R2)GO TO 10
IF(R3.GE.R2)GO TO 11
IF(R3.GE.R1)GO TO 12
J1=2
J2=1
J3=3
GO TO 15
10 J1=2
J2=3
J3=1
GO TO 15
11 J1=3
J2=2
J3=1
GO TO 15
15 CONTINUE.
RT1=W1(J1)
RT2=W1(J2)
RT3=W1(J3)
RTUHN
FNJ
SUBROUTINE DPROD(XR, XI, YR, YI, ZR, ZI)
DOUBLE PRECISION XR, XI, YR, YI, ZR, ZI
ZR = XR*YR - XI*YI
ZI = XI*YR + XR*YI
RETURN
END

SUBROUTINE DARCTN(YX, X, Z)
C FINDS DOUBLE PRECISION ARCTANGENT IN RADIANS IN CORRECT QUADRANT.
C DEFINES ZERO/ZERO=ZERO, USES RANGE PI TO - PI
DOUBLE PRECISION X, Y, Z, PI, YX
COMMON PI
YX = Y/X
300 IF(X)300,301,302
301 IF(Y)303,304,304
303 Z = DATAN(YX) - PI
GO TO 308
304 Z = DATAN(YX) + PI
GO TO 308
305 IF(Y)305,306,307
306 Z = PI/2.0
GO TO 308
307 Z = PI/2.0
GO TO 308
308 RETURN
END

SUBROUTINE BREIT(XI, FFF, XM, GJ, GI, DELW, RGAUSS, W, XGAUSS, EPS1)
ISGN = FFF
F = ISGN
SGN = (F*2.0 - 1.0)*(DELW/2.0)
GJ1 = GJ + 0.0029*(GJ - 1.0)
EPS1 = 1.0/(GJ1*1836.1/GI - 1.0)
XGAUSS = GJ1*9.2732*RGAUSS/(6.625*DELW*(1.0*EPS1))
EPS1 = EPS*DELW*XGAUSS
W = DELW/(4.0*XI + 2.0)*EPS1*XM
R = 2.0*XM/(XI + 0.5)
IF(R > 1.0)1,1,2
1 IF(XGAUSS - 1.0)2,3,3
3 SGN = SGN
2 W = W + SGN*SORT(1.0 + B*XGAUSS*XGAUSS + XGAUSS*XGAUSS)
IF(GJ.GE.1.0) RETURN
DELTA = ABS(W) + 4.0*EPS1*4214.0
W = W - DELTA
RETURN
END
APPENDIX B. A COMPUTER PROGRAM FOR THE NUMERICAL INTEGRATION OF THE FOUR-LEVEL EQUATIONS

A FORTRAN IV program which numerically integrates the four-level equations is given in this appendix. The version given finds |a|^2, |b|^2, |c|^2, |d|^2, and |a|^2 + |b|^2 as a function of time for a set of fixed driving frequencies and fixed magnetic field. The real and imaginary parts of a, b, c, and d are also given. Other versions of the program exist in which time is held constant while the variation of the above quantities with magnetic field at fixed frequency, or with frequency at fixed magnetic field, is studied.

On a CDC 6600, the program requires (for the accuracy used here) about 1 minute of central processor time per microsecond of integration time. More precisely, the computation times and number of times the subroutine DERIV is called are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>DERIV Calls per usec</th>
<th>Computation Time per usec</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5,000</td>
<td>35 sec</td>
<td>3 level, e^{i\omega t}</td>
</tr>
<tr>
<td>2</td>
<td>80,000</td>
<td>66 sec</td>
<td>3 level, cos \omega t</td>
</tr>
<tr>
<td>3</td>
<td>40,000</td>
<td>42 sec</td>
<td>4 level, e^{i\omega t}, e^{-i\omega t}</td>
</tr>
<tr>
<td>4</td>
<td>80,000</td>
<td>81 sec</td>
<td>4 level, cos \omega t</td>
</tr>
</tbody>
</table>

where the case number is as given in Figure 20. The program is believed to maintain better than 1% accuracy for integration times at least up to 1 microsecond. The accuracy can be adjusted with the parameters RELTST and ABSTST in subroutine INTEG. Accuracy testing is done in subroutine ACCRY.

The program is set up to allow the transverse electric field to oscillate also. This case could be of interest if the metastable beam was mixed with a plasma, since rf fields could penetrate the plasma, under appropriate conditions, while static electric fields cannot.

The input for the program is as follows:

Card 1 (FORMAT F12.6)  
Identical to Card 1 for program described in Appendix A.

Card 2 (FORMAT F12.6)  
Identical to Card 2 for program described in Appendix A.

Card 3 (FORMAT F12.6)  
FREQ2 driving transverse frequency in MHz (normally zero)

Card 4 (FORMAT F12.6)  
Identical to Card 3 for program described in Appendix A.

Card 5 (FORMAT F12.6)  
XMP F' in MHz (angular frequency)  
RMP R' in MHz (angular frequency)  
VMP V' in MHz (angular frequency)

where these quantities are complex and again require two numbers each for their specification.

Cards 6, 7 (FORMAT F12.6)  
X0(1) initial real part of a  
X0(2) initial imaginary part of a  
X0(3) initial real part of b  
X0(4) initial imaginary part of b  
X0(5) initial real part of c  
X0(6) initial imaginary part of c  
X0(7) initial real part of d  
X0(8) initial imaginary part of d

Card 8 (FORMAT F12.6)  
Identical to Card 4 of program described in Appendix A.

The program consists of a main program together with several subroutines. The function of the various subroutines is as follows:

a) SUBROUTINE INTEG (NN, TI, TTF, HH, HSP, MM, VVM, IP, X0, TT, XXP)  
This subroutine integrates an arbitrary system of real linear differential equations. The arguments of this subroutine are defined by comments in the main program listing. The monitoring feature (a periodic test of a specified variable against some
(limit) is not used. INTEG calls a number of sub-
routines of which only those which are specific to
the problem under discussion will be further de-
scribed.

b) SUBROUTINE DERIV (T, V, FD)
This subroutine computes the value of the first de-
rivatives FD(I) (for I = 1 to 8) given the value of
the functions V(I) (for I = 1 to 8) and the time T.
I = 1 and 2 correspond to the real and imaginary
parts of a, 3 and 4 to the real and imaginary parts
of b, and so on. (The four complex first-order dif-
ferential equations have been rewritten as eight
real first-order differential equations.)

c) SUBROUTINE PRINT (T, V)
This subroutine sets up the common variable arrays
at the specified print-step intervals for later
printout.

d) SUBROUTINE BREIT (XI, FFF, XM, GJ, GI, DELW,
BGAUSS, W, XGAUSS, EFG1)
This subroutine was described in Appendix A.

The output from the program is fully labeled. Tape
10 is defined as input and Tape 9 as output for the
system for which this program was written.

The inclusion of fields whose strength varies as
time (or displacement in a cavity) can be easily
incorporated in the subroutine DERIV. It is re-
quired to give RR, RRP, MM, MMP, VV, and VVP the
required time dependence, as indicated on comment
cards.

It is important to simplify DERIV as much as pos-
sible, from the point of view of computation time,
since it is in the innermost loop. The form listed
here is more general than required for many prob-
lems, and, if computer time is important, it should
be simplified in those cases.
READ(10,1)XSPIN,GJ,XMU,DELW,GJP,DELWP
READ(10,1)FREQMN,FREQDL,FREQMX,TTF,HP,BGAUSS
READ(10,1)FREQ2
GI=XMU/XSPIN
ISPIN=XSPIN*1.0
GO TO (100+101),ISPIN

100 NCASE=2
FFA(1)=1.0
XMFA(1)=1.0
FFB(1)=0.0
XMFB(1)=0.0
FFA(2)=1.0
XMFA(2)=0.0
FFB(2)=1.0
XMFB(2)=-1.0
GO TO 20

101 NCASE=3
FFA(1)=1.5
XMFA(1)=1.5
FFB(1)=0.5
XMFB(1)=0.5
FFA(2)=1.5
XMFA(2)=-0.5
FFA(3)=1.5
XMFA(3)=-0.5
GO TO 20

20 READ(10,1)XMM,RR,VV
READ(10,1)XMM,RR,VV
READ(10,1)(XO(I),I=1,8)
READ(10,13)IMODE,ICSMN,ICSMX
NFREQ=(FREOMX-FREQMN)/FREQDL
NFREQ=NFREQ+1
DO 110 ICASE=ICSMX,ICSMX
IITER=0
ITIME=0
CALL BREIT(XSPIN,FFA(ICASE),XMFA(ICASE),GJ,GJ,DELW,BGAUSS,FA,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFB(ICASE),XMFB(ICASE),GJ,GJ,DELW,BGAUSS,FB,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFA(ICASE),XMFA(ICASE),GJ,GJ,DELW,BGAUSS,FF,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFB(ICASE),XMFB(ICASE),GJ,GJ,DELW,BGAUSS,FE,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFA(ICASE),XMFA(ICASE),GJ,GJ,DELW,BGAUSS,FA,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFB(ICASE),XMFB(ICASE),GJ,GJ,DELW,BGAUSS,FB,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFA(ICASE),XMFA(ICASE),GJ,GJ,DELW,BGAUSS,FF,
1XGAUSS,EPSI)
CALL BREIT(XSPIN,FFB(ICASE),XMFB(ICASE),GJ,GJ,DELW,BGAUSS,FE,
1XGAUSS,EPSI)
FE=FE-1.058,070
FF=FF-1.058,070
FAB=FA-FB
FAE=FA-FA
FAF=FA-FF
FBE=FB-FE
FF=FBE-FF
FE=FE-FF
WAB=FAB*6.2831854
WAE=FAE*6.2831854
WAF=FAB*6.2831854
WBE=FB*6.2831854
WBF=FF*6.2831854
WEF=FF*6.2831854
FSP(ICASE)=FAB
FSP(ICASE+2)=FAE
FSP(ICASE+3)=FAF
FSP(ICASE+4)=FBE
FSP(ICASE+5)=FBE
FSP(ICASE+6)=FFE
DO 112 IFREQ=1,NFREQ
IFREQ=IFREQ
FREQ(IFREQ)=(IFREQ-1.0)*FREQDL+FREQMN
FREQ=IFREQ(IFREQ)
WFREQ=+2831854*FREQ1
WFREQ=+2831854*FREQ2
SUBROUTINE INTEG(NN, TI, TTF, HH, HHP, MM, VVM, XP, XO, TT, XXP)
C INTEG() SOLVES A SYSTEM OF N FIRST ORDER DIFF. EQUATIONS BY A 4TH
C ORDER ADAMS P-C METHOD WITH AUTOMATIC ERROR CONTROL. STARTING
C IS BY Runga-Kutta.
C
C SET UP INITIAL VALUES
N = NN
TF = TTF
HH = HH
HP = HHP

...
VM=VVM
DO 10 I=1,N
10 X(I)*X(I)
IF (P.EQ.0) GO TO 21
L=M+1
J=M+P
DO 20 I=L+U
XP(I)*X(I)
DO 20 J=1,5
20 X(I*J)*X(I)
21 T=U
B=M+1
ABS*B=1.*E-4
REL=ABS
ABSTST=ABS*K1.2
RELST=REL*K1.2
FACTOR=REL/ABS
LB=SP.00%*RELST
TUMU=M
NDM=2.*E3
H=2.*E3
30 CALL START(I'RETRN)
GO TO (100,99)IRETRN
C SHOULD ANY OF THE STARTING VALUES BE PRINTED OUT
100 T=T-3.*O*H
DO 35 J=2,4
TX=H
CALL TEST(I'RETRN)
GO TO (35,60)IRETRN
35 CONTINUE
C BEGIN ADAMS METHOD
40 CALL ADAMS
CALL ACCRY
IF (ACC) GO TO 50
DO 45 I=1,N
45 X(I)*X(I)
GO TO 30
50 CALL TEST(I'RETRN)
GO TO (101,60)IRETRN
101 CALL DOUBLE(I'RETRN)
GO TO (40,30)IRETRN
60 IF (J.EQ.5) GO TO 65
DO 64 I=1,N
64 XP(I)*X(I)
65 CALL PRINT(T*XP)
70 T=T
DO 70 I=1,N
70 XP(I)=XP(I)
RETURN
END
SUBROUTINE START (IRETRN)
C RUNG-KUTTA STARTING METHOD
LOGICAL ACC
COMMON/RLK1/N,T,T,F,H,H0,MP,H,VVM,M,J,ACC,RELST,ABSTST,FACTOR,
HND*X(3*5)+F(30,5),E(30)+XP(30)
COMMON/RLK2/G(30,4)
J=2
CALL RNGA
10 DO 15 I=1,N
15 XP(I)*X(I)
C XP(I)=UBL INTERVAL RESULT FOR ERROR ANALYSIS
T=T
H=H5*E
IF (T+W.0,E1) GO TO 30
W01 (9,20)
20 FORMAT(50H EUNs CANNOT BE SOLVED FURTHER WITHIN GIVEN ERROR )
53
**SUBROUTINE RNGA**

C INTEGRATE N EQUATIONS AHEAD ON THE J/TH STEP OF LENGTH H.

COMMON/RLKI/N*TQTF.H*HO*HP*MOVMcJ,ACCsLBtRELTST~ABSTST,FACTOR,
1BND*X(30,5),F(30,5),E(30),XP(30)
COMMON/RLK2/G(30,4)
CALL DERIV(T*X(I,J-1),F(I,J-1))
DO 10 I=1,N
G(I,J)=H*F(I,J)
10 X(I,J)=X(I,J-1)*0.5*(G(I,J)+G(I,J+1))
CALL DERIV(T*T*X(I,J),F(I,J))
DO 20 I=1,N
G(I,J+1)=H*F(I,J)
20 X(I,J+1)=X(I,J)+0.5*(G(I,J)+G(I,J+2))
CALL DERIV(T*T*X(I,J),F(I,J))
DO 30 I=1,N
G(I,J+2)=H*F(I,J)
30 X(I,J+2)=X(I,J)+0.5*(G(I,J)+G(I,J+3)+G(I,J+4))
RETURN
END

**SUBROUTINE ACCRY**

C TESTS ABS AND REL ERROR AND SETS ACC *FALSE* IF NEITHER SATISFIED

LOGICAL ACC
COMMON/BLK1/N*TQTF.H*HO*MP*M*VM*J,ACCsLBtRELTST~ABSTST,FACTOR,
1BND*X(30,5),F(30,5),E(30),XP(30)
ACC=TRUE
DO 50 I=1,N
E(I)=ABS(XP(I))
IF (E(I) .GE. ABS(X(I,J))*RELTST) GO TO 10
E(I)=E(I)/ABS(X(I,J))
50 CONTINUE
10 IF (E(I) .GE. ABSTST) GO TO 20
E(I)=E(I)*FACTOR
GO TO 50
20 T=T-H
H=0.5*H
ACC=.FALSE.
75 FORMAT(1H1,16HSTEP SIZE CUT TO, F16.8, 6H AT T, F12.8)
WRITE(9,75) H, T
GO TO 99
50 CONTINUE
99 RETURN
END
SUBROUTINE TEST (IRETNR)
C MONITORS FOR VM LIMITS OF INTEGR
COMMON /RLK1/N+TF,H+HP,M+VM,J+ACC*LB,H+RELST+ABSTST+FACTOR,
18ND*X(30+5)+E(30+5)+XP(30)
DIMENSION X1(30),X2(30),F1(30),F2(30)
IF (M.EQ.0) GO TO 20
IF (X(M+J-1)+GT+VM) GO TO 10
IF (X(M+J-1)+LE+VM) GO TO 10
GO TO 20
10 CALL DIODE
IF (T+TF) 70+70+30
70 IRETNR=2
RETURN
20 IF (ABS (T-TF)/TF-1.0E-6) 80,81,81
80 IRETNR=2
RETURN
81 IF (T+LE+TF) GO TO 40
30 H=TF-T
35 IF (T+LE+TF) GO TO 40
J=1
CALL RNGA
RETURN
40 IF (I+LT+AND) GO TO 50
C SAVE ALL VARIABLES WHICH MAY BE MODIFIED IN PRINT PROCEDURE
H=SAVE
T=SAVE
J=SAVE
DO 45 I=1+N
X(I+1)=X(I+1)
X(I+2)=X(I+2)
F1(I)=F1(I)
F2(I)=F2(I)
45 X(I+1)=X(I+1)
J=2
H=HND-T
CALL RNGA
CALL PRINT(T*X(I+1))
BND=HND+HP
C RESTORE VARIABLES TO PROCEED
J=JSAVE
H=SAVE
T=SAVE
DO 46 I=1+N
X(I+1)=X(I+1)
X(I+2)=X(I+2)
F1(I)=F1(I)
46 F(I+2)=F2(I)
50 IF (J+NE+5) GO TO 99
DO 60 I=1+N
X(I+1)=X(I+1)
DO 60 J=2+5
60 F(I+J-1)=F(I+J)
99 IRETNR=1
RETURN
END
C CALL DIODE
C FIND VALUE OF T WHERE THE M/TH VARIABLE REACHES THE VALUE VM
COMMON /RLK1/N+TF,H+HP,M+VM,J+ACC*LB,H+RELST+ABSTST+FACTOR,
18ND*X(30+5)+E(30+5)+XP(30)
DIMENSION D(30)
Y1=X(M+J)
Y0=X(M+J-1)
DELT=ABS (D*Y1/(Y1-Y0))
10 H=DELT
DO 20 I=1+N
20 X(I+1)=X(I+1)
J=*2
CALL RNGA
CALL DERIV(T*X(1,J),F)
DELT=(VM-X(M+J))/D(M)
IF (ABS(DELT) .GE.1.0E-4) GO TO 10
X(M+J)=VM
RETURN
END

SUBROUTINE ADAMS
C INTEGRATE ONE STEP BY THE ADAMS PREDICTOR-CORRECTOR METHOD
COMMON/HLK1,N,T,TF,H,H0,HP,H,M,V,M,J,ACC,LB,RELTST,ABSTST,FACTOR,
18NM,X(3N+5),F(30,5),E(30),XP(30)
J=*5
CALL DERIV(T*X(1,4)+F(1,4))
DO 10 I=1,N
10 XP(I)=X(I)+0.041666667*H*(55*0*F(I+1)-59*0*F(I+3)
1+37*0*F(I+2)-9*0*F(I+1))
RETURN
END

SUBROUTINE DOUBLE (IMETRN)
C CAN INTERVAL BE DOUBLED
REAL LH
COMMON/HLK5/I,DUP,LH,N,T,TF,H,H0,HP,H,M,V,M,J,ACC,LB,RELTST,ABSTST,FACTOR,
18NM,X(3N+5),F(30,5),E(30),XP(30)
COMMON/HLK5/IDOUBL,N
IDOUBL=IDOUBL+1
IF (IDOUBL.LT.NDIOUBL) GO TO 99
C ALLOWS DOUBLE ATTEMPT ONLY EVERY NDIOUBL/TH CALL
IDOUBL=n
DO 10 I=1,N
IF (E(I).GT.LB) GO TO 99
CONTINUE
10 D1=HP/(2.0*H)
IF (D1.LE.2.0) GO TO 99
D2=(HND-T)/(2.0*H)
IF (D2.LE.2.0) GO TO 99
DO 20 I=1,N
20 X(I)=X(I)+0.041666667*H*(9.0*F(I)+19.0*F(I+1)
1-5.0*F(I+2)+F(I+3))
RETURN
END

SUBROUTINE DERIV(T,V*FD)
DIMENSION V(30),FD(30)
COMPLEX UBA*UFEB*UEA*UFBR*UFAB*UEB
COMPLEX XMM*XMP*RRHRP*VW*VVP
COMMON/BLK3/XMM,XMP,RRHRP,VW*VVP,GAMMA,WAB*WAE*WAF*WBE*WBF*WEF,
1WFREQ1*WFREQ2

56
COMMON/RLK71IDERIV
IDERIV=IDERIV+1
FOFT1=CN5(WFREQ1*T)

C REPLACE FOFT1 BY FOFT1*(DESired SLOW FUNCTION OF TIME) TO MODULATE
C FOFT2=CN5(WFREQ2*T)

C IF FREQ2=0.0 IS ONLY CASE OF INTEREST SET FOFT2=1.0 TO SAVE COMPUTER
C TIME
C REPLACE FOFT2 BY FOFT2*(DESired SLOW FUNCTION OF TIME) TO MODULATE
C VV*VVP

UR=SN(WAE*T)
UI=COs(WAE*T)
UEA=RR*FOFT1*CMPLX(UR,UI)
UR=SN(WBE*T)
UI=COs(WBE*T)
UER=VV*FOFT2*CMPLX(UR,UI)
UER=REAL(UEA)
UERR=REAL(UER)
UEA=AIMAG(UEA)
UERR=AIMAG(UER)
FU(1)=UERA*V(5)-UERAI*V(6)
FU(2)=UERAI*V(5)-UERA*V(6)
FU(3)=UERAI*V(6)-UERA*V(5)
FU(4)=UERR*V(5)-UIERR*V(6)
FU(5)=UERA*V(1)+UEAI*V(2)+UEAI*V(3)+UERAI*V(4)+GAMMA2*V(5)
FU(6)=UERAI*V(1)+UERA*V(2)+UERAI*V(3)+UERA*V(4)+GAMMA2*V(6)
FU(7)=GAMMA2*V(7)
FU(8)=GAMMA2*V(8)

IF (CABS(RRP).NE.0.0) GO TO 101
IF (CABS(VVP).NE.0.0) GO TO 101
GO TO 200

101

1010 UR=SN(WBF*T)
UI=COs(WBF*T)
UFAR=RRP*FOFT1*CMPLX(UR,UI)
UR=SN(WAE*T)
UI=COs(WAE*T)
UFAR=VVP*FOFT2*CMPLX(UR,UI)
UFAR=REAL(UFAR)
UFAR=AIMAG(UFAR)
FU(1)=UFAR*V(1)-UFAR*V(2)
FU(2)=UFAR*V(2)-UFAR*V(3)
FU(3)=UFAR*V(3)-UFAR*V(4)
FU(4)=UFAR*V(4)-UFAR*V(5)
FU(5)=UFAR*V(5)-UFAR*V(6)
FU(6)=UFAR*V(6)-UFAR*V(7)
FU(7)=UFAR*V(7)-UFAR*V(8)
FU(8)=UFAR*V(8)-UFAR*V(9)

IF (CABS(RRP).NE.0.0) GO TO 201
IF (CABS(XMM).NE.0.0) GO TO 201
GO TO 300

201

2010 UH=SN(WAF*T)
UI=COs(WAF*T)
UFA=XMM*FOFT1*CMPLX(UR,UI)
UR=SN(WBE*T)
UI=COs(WBE*T)
UFA=XMM*FOFT2*CMPLX(UR,UI)
UFA=REAL(UFA)
UFA=AIMAG(UFA)
FU(1)=UFA*V(1)-UFA*V(2)
FU(2)=UFA*V(2)-UFA*V(3)
FU(3)=UFA*V(3)-UFA*V(4)
FU(4)=UFA*V(4)-UFA*V(5)
FU(5)=UFA*V(5)-UFA*V(6)
FU(6)=UFA*V(6)-UFA*V(7)
FU(7)=UFA*V(7)-UFA*V(8)
FU(8)=UFA*V(8)-UFA*V(9)

IF (CABS(XMM).NE.0.0) GO TO 201
IF (CABS(XMM).NE.0.0) GO TO 201
GO TO 300

300 RETURN
END
SUBROUTINE PRINT(T,V)
DIMENSION V(30)
COMMON ARL(3,100,5),BRL(3,100,5),CRL(3,100,5),DRL(3,100,5),
LAIM(3,100,5),RIM(3,100,5),CIM(3,100,5),DIM(3,100,5),ICASE,IFREQ,
ITIME,TIME(100)
COMMON/BLK1/N,T,F,HM,HP,H,M,VM,J,ACC,AL,HSTST,ASTST,FACTOR,
BNDX(3n,5),E(30),AP(30)
COMMON/BLK7/IDERV
ITIME=ITIME+1
ITIME=ITIME
WRITE(9,10) ICASE,ITIME,IFREQ,DERIV
10 FORMAT(6H ICASE 14,6H ITIME 14,6H IFREQ 14,13H DERIV CYCLES I10)
11 FORMAT(9F12.6)
WRITE(9,11)T*V(I) I=1,8
APL(ICASE,ITIME,IFREQ)=V(1)
ARL(ICASE,ITIME,IFREQ)=V(2)
BRL(ICASE,ITIME,IFREQ)=V(3)
BIM(ICASE,ITIME,IFREQ)=V(4)
CRL(ICASE,ITIME,IFREQ)=V(5)
OIM(ICASE,ITIME,IFREQ)=V(6)
CIM(ICASE,ITIME,IFREQ)=V(7)
DIM(ICASE,ITIME,IFREQ)=V(8)
RETURN
END

SUBROUTINE HHEIT(XI,FFF,XM,GJ,DELW,BGAUSS,WM,XGAUSS,EP51)
ISGN=FFF
F(ISGN)
SGN=(F=0.0-1.0)*(DELW/2.0)
GJ=GJ+0.0229*(GJ-1.0)
EPS=1.0/(GJ*1836.1/GJ-1.0)
XGAUSS=GJ*9.2732*FGAUS/ (6.625*NELW*(1.0+EPS))
5 EPS1=EPS*DELW*XGAUSS
6 W=DELW/(4.0*XI+0.0)*EPS1*XM
R=2.0*XM/(XI+0.5)
IF(R+1.0) I=1
1 IF(XGAUSS-1.0)2033
3 SGN=SGN
2 W=W+SGN*SORT(1.0+B*XGAUSS*XGAUSS*XGAUSS*XGAUSS)
IF(GJ.GE.1.0) RETURN
DELTA=ABS(W)*4.0*BGAUSS/(9.0*5214.0)
W=W+DELTA
RETURN
END