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Bethe-Salpeter Equation: Numerical Experience with a Hydrogenlike Atom

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Abstract

The eigenvalue $\lambda$ of the Bethe-Salpeter equation,

$$\lambda \phi = \frac{1}{\pi^2} G S \phi,$$

with the value of $E_b$ calculated from the Sommerfeld expression,

$$E_b = \frac{1}{2} m c^2 \left( 1 - \frac{1}{\sqrt{1 + (1/137.03802)^2}} \right),$$

substituted into the propagator $S$, and only the one-photon-exchange term included in $G$, is, for the $^1S_0$ state,

$$\lambda = 133.9963 \pm 0.0003.$$

The difference between $\lambda$ and 137.03802 is due to the appearance of a logarithmic term in

$$E_b = \frac{1}{4} m c^2 \alpha^2 \left( 1 - \frac{4\alpha}{\pi} \ln \frac{2}{\alpha} + \ldots \right),$$

which is valid for the Bethe-Salpeter equation, $^1S_0$ state, one photon exchange. The crossed two photon exchange contribution brings $\lambda$ up to

$$\lambda = 136.5 \pm (0.1?),$$

so that this contribution cancels the logarithmic term, but not exactly.
I. INTRODUCTION

In a paper in Physical Review with this same title, we reported that the binding energy $E_b$ of an equal mass $m$ hydrogen atom resulting from the Bethe-Salpeter equation with one photon exchange differs from the $E_b$ resulting from the Dirac equation with a Coulomb potential in that logarithmic terms occur:

$$E_b = \frac{1}{4} mc^2 \alpha^2 \left(1 - \frac{4\alpha}{\pi} \ln \frac{2}{\alpha} + \ldots\right).$$

The numerical results substantiating the analytic work were expressed in the following way. The Bethe-Salpeter equation is

$$\lambda \phi = \frac{1}{\pi^2} GS \phi,$$

where $\phi$ is the Bethe-Salpeter amplitude, $G$ the sum of all two particle irreducible graphs (in these calculations only the one photon exchange term is included), $S$ the direct product of the fermion propagators, and $\lambda = 4\pi/e^2 = 1/\alpha$ is a factor removed from the one photon exchange graph (for more details see reference 1). $S$ contains $E_b$, and we put in the value calculated from the Sommerfeld expression

$$E_b = \frac{1}{2} mc^2 \left(1 - \frac{1}{\sqrt{1 + \alpha^2}}\right),$$

with the accepted value $\alpha = 1/137.03802$. We find no difference between the Bethe-Salpeter $E_b$ and the Dirac $E_b$ we should then find the eigenvalue $\lambda = 137.03802$. We do not find agreement because of the logarithmic term in Eq. (1).

In ref. 1 we reported $\lambda = 134.18 \pm 0.01$, and in this work we find $\lambda = 133.9963 \pm 0.0003$ by the use of improved numerical techniques. Our
motivation for making this improvement is the fact that should we want to calculate the Lamb shift by these techniques, such accuracy is required. Our general method, which is a straightforward solution of the Bethe-Salpeter equation, is conceptually simpler than the usual methods, but not yet nearly so accurate or complete.

We would like to explain a little more our motivation. We have calculated $\lambda$ with the crossed two photon exchange graph included in $G$. We find $\lambda = 136.5$. Thus our results underscore Salpeter's remark that the calculation done in a straightforward manner converges very slowly as more and more graphs are included in $G$. In calculating the Lamb shift it is not necessary to include the crossed two photon exchange graph and similar graphs because the ladder approximation yields a spectrum with the same degeneracy as the Dirac equation with a Coulomb potential. Only the level splitting resulting from the self energy graph shown in Fig. 1 is desired, and this splitting may be calculated by including in $G$ the one photon exchange graph and the self energy graph. The matrix elements of the self energy graph may be read from the book by Jauch and Rohrlich since they give the full off-shell matrix elements required (Jauch and Rohrlich give the part arising from the bubble in the fermion line and this has to be multiplied by the one photon exchange part which is easily calculated by the methods of reference 1). Our interest in doing the calculation in this way arises from our belief that the infrared problem does not arise. We believe this because the bound state problem is off-shell and it may be observed that Jauch and Rohrlich's result has no divergence off shell. As the binding energy goes to zero, the problem moves toward the on-shell case, so terms like $\log (1/\alpha)$ should occur. We wonder how these logarithmic terms cancel against the one shown in Eq. (1), which according to the result $\lambda = 136.5$ is only partially cancelled by the crossed two photon exchange
part, and it is hard to see how the crossed three photon exchange parts will make up the difference. Perhaps it is precisely the self energy graph which makes up the difference (the photon self energy graph also has a part to play). We hasten to add in conclusion that we have not yet done the calculations, but the calculations are possible (Jochem Fleischer has done similar calculations in the strong interaction case\textsuperscript{7,8}). We may hope not to remain forever ignorant of the facts. Many\textsuperscript{9} are aware of just how awkward the usual treatment of the infrared problem is, and if it is true that it does not really arise in a full but straightforward approach, it would be a matter of interest.

II. THE IMPROVED NUMERICAL TECHNIQUES

We restrict the calculation to the positive energy states, so that

\[ \lambda \phi(p, ip_4) = \frac{1}{\pi^2} \int_{0}^{\infty} dq \int_{0}^{\infty} dq_4 K(p, ip_4, q, iq_4) S(q, iq_4) \phi(q, iq_4) \]  

(4)

The numerical approach to any such equation is to replace the integrations by finite sums over a mesh. Symbolically,

\[ \int dq \int dq_4 = \Sigma + \int f - \Sigma f + \Sigma f - \Sigma \]

(5)

\[ = \Sigma + (f - \Sigma) f + \Sigma f - \Sigma \]
may be referred to as a correction for the $q$ integration and
$\Sigma(f - \Sigma)$ as a correction for the $q_4$ integration. In these corrections,
we do not need to use the exact $\phi(q, iq_4)$, which is unknown anyhow, but we
must keep the equations homogeneous. As proposed in reference 1, we make
use of the fact that $\phi(q, iq_4)$ is nearly independent of $q_4$, and in the
$q_4$ correction we use

$$\phi(q, iq_4) = \phi(q, 0).$$ \hspace{1cm} (6)

In the $q$ correction, we make use of this fact and the fact that we know
approximately how $\phi(q, 0)$ depends on $q$; namely, it must depend on $q$ in
the same way that the nonrelativistic $\phi(q)$ does. Thus in the $q$ correction
we use

$$\phi(q, iq_4) = \frac{q}{q_4} \frac{p^2}{2m + \frac{1}{2} \frac{E}{E_b}} \phi(p, 0).$$ \hspace{1cm} (7)

In detail, our finite sum approximation to Eq. (4) is

$$\pi^2 \lambda \phi(p, ip_4) = \Sigma \Sigma K(p, ip_4, q, iq_4) S(q, iq_4) \phi(q, iq_4)$$
$$+ T(p, ip_4) \phi(p, 0) + \Sigma R(p, p_4, q) \phi(q, 0),$$ \hspace{1cm} (8)

where

$$R(p, p_4, q) = \int_0^{\infty} dq_4 \left[ 0 \right] K(p, ip_4, q, iq_4) S(q, iq_4)$$
$$= R_1 - R_2,$$ \hspace{1cm} (9)

and
The integrals can be done analytically (otherwise there would be no point in writing them down) provided we use some nonrelativistic kinematics in $K$ and $S$, namely

$$K(p, ip_4, q, iq_4) = \frac{1}{4} \ln \left[ \frac{(p + q)^2 + (p_4 + q_4)^2}{(p - q)^2 + (p_4 + q_4)^2} \right].$$

$$S(q, iq_4) = \frac{1}{(\frac{q^2}{2m} + \frac{1}{2} E_b)^2 + q_4^2}. \quad (11)$$

Of course we use these "nonrelativistic" approximations only in corrections which vanish as the mesh size is refined. We find

$$R_1(p, p_4, q) = \frac{\pi}{4} \ln \left[ \frac{(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q)^2 + p_4^2}{(\frac{q^2}{2m} + \frac{1}{2} E_b + p - q)^2 + p_4^2} \right].$$

$$S_1(p, p_4) = f(p, i\sqrt{\frac{E_b}{m}}, ip_4) + f(p, i\sqrt{\frac{E_b}{m}}, -ip_4) - f(p, -i\sqrt{\frac{E_b}{m}}, ip_4) - f(p, -i\sqrt{\frac{E_b}{m}}, -ip_4).$$

$$= S_1 - S_2 \quad \cdots \quad (10)$$
\[
f(p, i \sqrt{\frac{E_b}{m}}, i p_4) = 137.03802 \frac{2}{3} \pi \frac{P_m^2 + \frac{1}{2} E_b}{P} \left[ F\left(\frac{P_m}{m}, 0, 1 \sqrt{\frac{E_b}{m}}\right) - F\left(\frac{P_m}{m}, 0, -1 - \sqrt{1\left(1\right)}\right) \right]
\begin{align*}
& - \frac{F\left(\frac{P_m}{m}, 0, 1 \sqrt{\frac{E_b}{m}}\right) - F\left(\frac{P_m}{m}, 0, 1 - \sqrt{1\left(1\right)}\right)}{i \sqrt{\frac{E_b}{m}} + 1 + \sqrt{1\left(1\right)}} \\
& + \frac{F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, 1 \sqrt{\frac{E_b}{m}}\right) - F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, -1 - \sqrt{1\left(2\right)}\right)}{i \sqrt{\frac{E_b}{m}} + 1 + \sqrt{1\left(2\right)}} \\
& - \frac{F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, 1 \sqrt{\frac{E_b}{m}}\right) - F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, 0 - \sqrt{1\left(1\right)}\right)}{i \sqrt{\frac{E_b}{m}} + 1 - \sqrt{1\left(1\right)}} \\
& + \frac{F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, 1 \sqrt{\frac{E_b}{m}}\right) - F\left(\frac{P_{\text{max}}}{m}, \frac{P_m}{m}, -1 + \sqrt{1\left(2\right)}\right)}{i \sqrt{\frac{E_b}{m}} + 1 - \sqrt{1\left(2\right)}}
\end{align*}
\]
\[ F(x_2, x_1, z) = \frac{1}{2} \log \left( w^2 + \left( \frac{z-z^*}{2i} \right)^2 \right) \]

\[ + \arctan \left( \frac{w}{z-z^*} \right) \]

\[ = \frac{x_2 - z + z^*}{2} - \frac{x_1 - z + z^*}{2} \]

\[ \frac{\pi}{2} > \arctan > -\frac{\pi}{2} \]

\[ \sqrt{(1)} = \sqrt{1 - \frac{2}{m} \left( \frac{1}{2} E_b - p + i p_4 \right)} \]

\[ \sqrt{(2)} = \sqrt{1 - \frac{2}{m} \left( \frac{1}{2} E_b - p + i p_4 \right)} \]

\[ p_{\text{max}} = \text{largest value of } p \text{ used in the } p_{\text{mesh}} \]

Methods by which these formulae were derived are outlined in Appendix I.

We transform everything to the \( u,v \) meshes from the \( p,p_4 \) meshes as in reference 1. The various \( \Sigma \) need to have their correct weight factors and the Jacobians of the transformations as described in reference 1.

Because the corrections are not exact, the results will still depend on mesh size and the parameters \( a \) and \( b \) introduced as corrections to integration schemes in reference 1 (see Eq. (20) of reference 1).

We have, of course, wondered why our error in the calculation reported in reference 1 is about 20 times larger than we thought. To find out, we have repeated the calculation of reference 1 with still finer mesh sizes. We see that \( d\lambda/ds \) (mesh size) = 0 for the \( a,b \) and the finest mesh sizes used in reference 1, and this led to the illusion that \( \lambda \) was
independent of mesh size to the stated accuracy for these meshes. But as
the calculations with still finer meshes show, it is only an illusion.
This danger always exists and can never be completely eliminated by
numerical means; it still exists for this improved calculation.

III. RESULTS

All results are gathered in Fig. 2 and Table I. The best extrapolated
value is

$$\lambda = 133.9963 \pm 0.0003$$

The danger referred to just above is reduced by the fact that we have
results approaching this limit from below and above, as shown in Fig. 2.

IV. CONCLUSION

We believe that we have shown that it is possible to solve the Bethe-
Salpeter equation by the use of straightforward numerical techniques to an
accuracy required by the precision measurement of atomic physics.
To evaluate $P_1$, expand the argument of the logarithm in Eq. (11) thusly:

\[
\left[ \right. = \frac{(p + q)^2 + p_4^2 + q_4^2)^2 - 4p_4^2 q_4^2}{((p - q)^2 + p_4^2 + q_4^2)^2 - 4p_4^2 q_4^2}
\]

\[
= \frac{(q_4^2 - r_1)(q_4^2 - r_2)}{(q_4^2 - r_3)(q_4^2 - r_4)}
\]

where $r_1, r_2, r_3,$ and $r_4$ depend on $p, q,$ and $p_4$. Using $\ln (AB/CD) = \ln A + \ln B - \ln C - \ln D$ leads to integrals of the form

\[
\int_0^\infty \frac{dq_4}{a^2 + q_4^2} \ln(q_4^2 + c^2) = \pi a \ln(a + c)
\]

The evaluation of $S_1$ requires the evaluation of

\[
\frac{1}{4\pi} \int_0^\infty \frac{q \, dq}{(\frac{q^2}{2m} + \frac{1}{2} E_b)^2} \ln \left( \frac{(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q)^2 + p_4^2}{(\frac{q^2}{2m} + \frac{1}{2} E_b + p - q)^2 + p_4^2} \right)
\]

\[
= \frac{m}{4\pi} \int_0^\infty \frac{q \, dq}{(\frac{q^2}{2m} + \frac{1}{2} E_b)^2} \ln \left( \frac{(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q)^2 + p_4^2}{(\frac{q^2}{2m} + \frac{1}{2} E_b + q - p)^2 + p_4^2} \right)
\]

\[
+ \frac{m}{4\pi} \int_0^\infty \frac{q \, dq}{(\frac{q^2}{2m} + \frac{1}{2} E_b)^2} \ln \left( \frac{(\frac{q^2}{2m} + \frac{1}{2} E_b + p + q)^2 + p_4^2}{(\frac{q^2}{2m} + \frac{1}{2} E_b + q - p)^2 + p_4^2} \right)
\]
\[
\frac{x}{x^2 + A^2} = \frac{1}{2} \left[ \frac{1}{x + 1A} + \frac{1}{x - 1A} \right],
\]

\[x = \frac{q^2}{2m} + \frac{1}{2} E_b + p + q\]

\[A = p_4\]

\[\frac{1}{x^2 + B^2} = \frac{1}{2IB} \left[ \frac{1}{x - iB} - \frac{1}{x + iB} \right],
\]

\[x = q\]

\[B = \sqrt{m E_b}\]

where the last equality results from an integration by parts. We then proceed by breaking various expressions into partial fractions; for example, in the first term on the right of the preceding equation,
and in this way we arrive at the expression for $S_1$ given in the text in terms of four $f$ functions, where

$$f = \frac{1}{\sqrt{mE_b}} \frac{m^2}{4\pi i} \left\{ \int_0^{p_{\text{max}}} \frac{dq}{q - i\sqrt{mE_b}} \left[ \frac{q}{2m} + \frac{1}{2} E_b + p + q + ip_4 \right. \right.$$

$$\left. \left. \left. \frac{q - 1}{q + 1} \right] \right\}$$

The four denominators quadratic in $q$ have roots which depend on $p, p_4$.

$$r_1 = r_3 = \left[ -1 \pm \sqrt{(1)} \right] m$$

$$r_2 = \left[ 1 \pm \sqrt{(1)} \right] m$$

$$r_4 = \left[ -1 \pm \sqrt{(2)} \right] m$$

where $\sqrt{(1)}$ and $\sqrt{(2)}$ are given in the text. Further reduction by partial fractions leads to
\[
\sqrt{\frac{m}{E_b}} \left\{ \frac{P_m}{\mathcal{I}} \right\} \int_0^1 d\left(\frac{q}{m}\right) \left[ \frac{1}{q - 1 - \frac{E_b}{m} \sqrt{\frac{1}{m} + \sqrt{1}}} \right] \frac{1}{1 - \frac{E_b}{m} + \sqrt{1}}
\]

\[
- \left[ \frac{1}{q - 1 - \frac{E_b}{m} \sqrt{\frac{1}{m} + \sqrt{1}}} \right] \frac{1}{1 - \frac{E_b}{m} - 1 + \sqrt{1}}
\]

\[
\int \frac{P_{\text{max}}}{m} d\left(\frac{q}{m}\right) \left[ \frac{1}{q - 1 - \frac{E_b}{m} \sqrt{\frac{1}{m} + \sqrt{2}}} \right] \frac{1}{1 - \frac{E_b}{m} + \sqrt{1}}
\]

\[
- \left[ \frac{1}{q - 1 - \frac{E_b}{m} \sqrt{\frac{1}{m} + \sqrt{2}}} \right] \frac{1}{1 - \frac{E_b}{m} - 1 + \sqrt{1}}
\]

+ a similar expression with \(-\sqrt{1}\) and \(-\sqrt{2}\), which explains the source of the sixteen F functions in the text. The F are all of the form

\[
F(x_2, x_1, z) = \int \frac{x_2}{x_1} \frac{dx}{x - z}
\]

and the result broken into obviously real and obviously imaginary parts is given in the text.
In this appendix we document the very complicated calculations including the crossed two photon exchange graph.

The crossed two pion exchange graph for nucleon-nucleon scattering has been evaluated independently by at least two groups\(^\text{10}\) and the algebra has been checked by use of the algebraic manipulating code REDUCE\(^\text{8}\). The result is that the positive energy-positive energy part of \( G \) is

\[
G(pp_0 qq_0) = \alpha \frac{pq}{E(p)E(q)} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 I(x_1, x_2, x_3, p, p_0, q, q_0).
\]

One factor of \( \alpha \) has been removed to include in the eigenvalue \( \lambda \). We have

\[
I = I_{0,2} T_1 + I_{1,2} T_2 + I_{0,1} T_3,
\]

\[
T_1 = E^+ (m^2 + P_1 P_2) + E^- z_{12} - E(p) (m^2 p^+ + q^2 b^-) - E(q) (m^2 p^+ + p^2 a^-) + m^2 (p^2 + q^2) + pq z_3,
\]

\[
T_2 = pq (- m^2 + P_1 P_2 + z_{12} + 2 m^2 - E(q) b^- - E(p) a^- + z_3 E^-),
\]

\[
T_3 = 2 E(p) E(q) - m^2,
\]

\[
P_1 = E(1 - x_1 - x_2) + (x_2 + x_3) q_0 + (1 - x_1 - x_3) p_0,
\]

\[
P_2 = E(1 - x_1 - x_2) - (1 - x_2 - x_3) q_0 - (x_1 + x_3) p_0,
\]

\[
E^- = E(p) E(q) - m^2,
\]

\[
E^+ = E(p) E(q) + m^2,
\]

\[
\sigma^- = E(1 - x_1 - x_2) - (1 - x_1 - x_2 - 2 x_3) q_0,
\]

\[
-14-\]
\[ b^- = E(1 - x_1 - x_2) + (1 - x_1 - x_2 - 2x_3)p_0 \]

\[ z_{12} = (x_1 + x_3)(1 - x_1 - x_3)p^2 + (x_2 + x_3)(1 - x_2 - x_3)q^2 \]

\[ z_3 = pq \{ (x_1 + x_3)(x_2 + x_3) + (1 - x_2 - x_3)(1 - x_1 - x_3) \} \]

\[ I_{j,1} = -\frac{1}{A} Q_j(s) \text{ for } A \neq 0, \quad j = 0, 1 \]

\[ I_{0,1} = \frac{1}{B}, \quad I_{1,1} = 0 \text{ for } A = 0 \]

\[ I_{j,2} = \frac{1}{A} \left[ \frac{s^j}{s^2 - 1} - j \frac{Q_0(s)}{A} \right] \text{ for } A \neq 0, \quad j = 0, 1 \]

\[ I_{0,2} = \frac{1}{B} \left( 1 - \frac{1}{s} - 0 \right) \text{ for } A = 0 \]

\[ s = -\frac{B}{A} \]

\[ A = 2pq \{ x_3(1 - x_1 - x_2 - x_3) - x_1x_2 \} \]

\[ B = x_1(1 - x_1) \{ (E + p_0)^2 - p^2 \} + x_2(1 - x_2) \{ (E - q_0)^2 - q^2 \} \]

\[ + x_3(1 - x_3) \{ (p_0 - q_0)^2 - p^2 - q^2 \} - 2(x_1x_2(E + p_0)(E - q_0) \]

\[ + x_1x_3 \{ (E + p_0)(p_0 - q_0) - p^2 \} + x_2x_3(E - q_0)(p_0 - q_0) - q^2 \]

\[ - \mu^2 (1 + x_1 + x_2) \]

\[ Q_0(s) = \frac{1}{2} \log \frac{s + 1}{s - 1} \]

\[ Q_1(s) = s Q_0(s) - 1 \]

\[ m = \text{fermion mass} \]

\[ \mu = \text{boson mass} \]

When the \( \gamma_5 \)'s appropriate to the pseudoscalar case are replaced by \( \gamma_\mu \)'s appropriate to the vector case, one would think that the above
expression is without relevance. However, the main part of the $\gamma_1$ vertex is $\gamma_0$ (this is the nonrelativistic limit and constitutes a very accurate approximation when we restrict ourselves to the positive-energy states).

Now consider a fermion line in the crossed two photon exchange graph (see Fig. 3). One of the $\gamma_0$'s may be brought through the propagator (as may one of the $\gamma_5$'s in the pseudoscalar case), and use made of $\gamma_0^2 = 1$ or $\gamma_5^2 = 1$. Since

$$p = \gamma_0 p_0 - \gamma_5 \cdot P,$$

bringing through the $\gamma_0$ or $\gamma_5$ is the same except that the sign of $p_0$ is changed in the $\gamma_5$ case and not in the $\gamma_0$ case; that is, the two cases differ only in the sign of $p_0$. In the calculations in Ref. 7 above, the external energy is carried by the nucleon line,

$$p_0 = k_0 + E,$$

where $k_0$ is a variable integrated over in the $\int d^4k$ required for the crossed two photon exchange graph. But the sign of $k_0$ does not matter since it is integrated over. One concludes that the pseudoscalar and vector cases (restricted to the $\gamma_0$ part) are the same except for the sign of $E$. Therefore,

$$E = -0.511 000698682,$$

$$m = 0.51100410,$$

$$\mu = 0.$$

For the rest, we make a number of transformations of the variables $x_1, x_2, x_3$. First,
\[ x_1 = zy \],

\[ x_2 = z(1 - y) \],

\[ x_3 = (1 - z)x \],

transforms

\[
\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 I = \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{1}{z(1 - z)} I.
\]

Then we transform

\[ z = \frac{z_{max} w}{1 - z_{max} - (1 - 2z_{max})w} \]

which transforms the integral to

\[
\int_0^1 dw \int_0^1 dx \int_0^1 dy \frac{z(I - z) z_{max}(1 - z_{max})}{[1 - z_{max} - (1 - 2z_{max})w]^2},
\]

in which \( z \) is a function of \( w \) of course.

For each \( p, p_0, q, q_0 \) we adjusted \( z_{max} \) so that the maximum value of the integrand occurred for \( \frac{3}{8} < w < \frac{5}{8} \). \( z_{max} \) varied from \( 2 \times 10^{-7} \) to 0.5 depending on the values of \( p, p_0, q, q_0 \). \( z_{max} = 2 \times 10^{-7} \) reflects how close to the boundary the peak of the integrand may occur. We use about 20 mesh points in the \( w \) integration and 5 each in the \( x \) and \( y \) integration.

This calculation has to be joined to the calculation described in Ref. 1. Everything is quite straightforward, and we believe that the above details sufficiently document the entire calculation leading to \( \lambda = 136.5 \). It was only possible to use the crudest \( q, q_0 \) mesh (with \( a = 1.377, b = 1.168 \)), which means already that 10,000 integrals of the above form have to be calculated, and the accuracy has to be judged from Fig. 2, plus worries about...
the w, x, and y meshes. We believe $\lambda = 136.5$ is good to $\pm 0.3$, but perhaps in the future with still faster computing machines it will be possible to verify this assertion.
References

* Work performed under the auspices of the United States Atomic Energy Commission.

3. The calculations are documented in Appendix II.
5. We assume this statement is true. We should be delighted to have a reference or a refutation.
9. See the statement in J. D. Bjorken and S. Drell, Relativistic Quantum Mechanics, McGraw Hill Book Company (New York), 1964, in the last paragraph at the bottom of p. 179.
Table I.

Data plotted in Fig. 2.

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1. The self energy graph.

2. $\lambda$ vs. q-mesh size. $192/x = \text{number of points in q-integration}$.
   
   (a) Results reported in Ref. 1. $a = 1.377$, $b = 1.168$, 16 point Gaussian $q_4$ integration.
   
   (b) $q_4$-correction applied to (a). Eight points used in Gaussian $q_4$ integration.
   
   (c) Both corrections applied. $a = 1.5$, $b = 0$, 8 point Gaussian $q_4$ integration.
   
   (d) Eoth corrections applied. $a = 1.377$, $b = 1.168$, 8 point Gaussian $q_4$ integration.

3. A portion of the two photon exchange graph.
Figure 1
\[ \frac{p + m}{p^2 - m^2} \]

Figure 3