1. LETTER FROM AUGUSTIN FRESNEL TO FRANÇOIS ARAGO, ON THE INFLUENCE OF THE MOVEMENT OF THE EARTH ON SOME PHENOMENA OF OPTICS*†

My dear friend,

By your fine experiments on the light from the stars, you have shown that the movement of the terrestrial globe has no perceptible influence upon the refraction of rays emanating from these stars. Within the corpuscular theory, as you have pointed out, this remarkable result can only be explained by supposing that luminous bodies transmit to the particles of light an infinite number of different velocities, and that these particles only affect the organ of sight when travelling at one of these velocities, or at least between very close-set limits, so that an increase or decrease of a ten-thousandth part is more than enough to prevent their detection. The necessity for this hypothesis is not the least difficulty attaching to the corpuscular theory; for on what does vision depend? Upon the impact of the light particles on the optic nerve? In this case such an impact would not be rendered imperceptible by an increase

* *Ann. de Chimie* **9**, 57 (1818).

† Taken from a letter to Léonor Fresnel, 5 September 1818 (LIX): ". . . I have recently been engaged on a small work to which I attach some importance. I have proved that, supposing the earth to be sufficiently porous to the ether which penetrates and surrounds it, not to transmit to it more than a minute part of its velocity, not exceeding, for example, an hundredth part, one could explain satisfactorily not only the aberration of the stars, but also all the other optical phenomena which are complicated by the movement of the earth, etc." (H. de St.).

in velocity. Upon the way in which the particles are refracted within the pupil? But red particles, for example, whose velocity had been diminished even by a fiftieth part, would still be refracted less than violet rays, and would not leave the spectrum which defines the limits of vision.

You have enjoined me to examine whether the result of these observations could be reconciled more easily with the theory in which light is considered as being vibrations of a universal fluid. It is all the more necessary to find an explanation within this theory, because the theory applies to terrestrial objects; for the velocity of wave propagation is independent of the movement of the body from which the waves emanate.

If one were to admit that our earth transfers its movement to the ether surrounding it, it would be easy to see why the same prism would always refract light in the same way, whatever direction it came from. But it appears impossible to explain the aberration of stars by this hypothesis: I have been unable, up to the present at least, to understand this phenomenon clearly except by supposing that the ether passes freely through the globe, and that the velocity communicated to this subtle fluid is only a small proportion of the velocity of the earth, not exceeding, for example, an hundredth part.

However extraordinary this hypothesis may appear at first sight, it does not seem to me at all incompatible with the idea of the extreme porosity of bodies which the greatest physicists have arrived at. It may indeed be asked how, while a very thin opaque body is capable of intercepting light, a current of ether can pass through our globe. While not claiming to meet this objection completely, I shall nevertheless point out that the two kinds of movement are too unlike in character for observations made in connection with one to be applicable to the other. The movement of light is not a current, but a vibration of the ether. One may see how the small elementary waves into which light divides when passing through a body may, in certain cases, be out of phase

on coming together again, by reason of the different paths they have taken or the different amounts by which they have been slowed down; this prevents the propagation of vibrations, or alters them in such a way as to remove their light-giving property, as occurs in a very striking manner with black bodies; while the same circumstances would not prevent the establishment of a current of ether. The transparency of hydrophane[†] is increased by wetting it, and it is evident that the interposition of water between its particles, while favouring the propagation of light vibrations, must on the contrary prove a small additional obstacle to the establishing of a current of ether. This is a good demonstration of the great difference which exists between the two types of movement.

The opacity of the earth is therefore not a sufficient reason to deny the existence of a current of ether between its molecules, and one may suppose it porous enough to communicate to this fluid only a very small part of its movement.

With the aid of this hypothesis the phenomenon of aberration is as easily explained by the theory of waves as by the corpuscular theory; for it arises out of the displacement of the optical instrument while the light is travelling through it. Now according to this hypothesis, the light waves do not participate to any perceptible extent in the movement of the telescope, which I am assuming to be pointed at the true location of the star, and the image of the star lags behind in relation to the cross-hair situated in the eyepiece of the telescope, by an amount corresponding to the distance covered by the earth while the light is travelling down the telescope.

It now remains to explain why, by the same hypothesis, the apparent refraction does not vary with the direction of the light rays relative to the movement of the earth.

† Hydrophane: a variety of opaque or partly translucent opal which absorbs water upon immersion and becomes transparent. [Translator's note.]

Let *EFG* (Fig. 1.1) be a prism, of which one side *EF* is placed at right angles both to the ecliptic and to the incident rays, which are thus travelling in the same direction as the earth: if the prism's movement has an influence upon their refraction, this is the case where it must be most apparent. I am supposing that the rays are moving in the same direction as the prism.

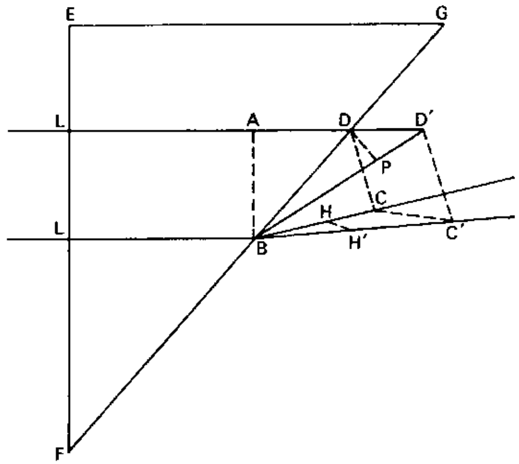

FIG. 1.1

Since they strike the surface of entry at right angles, the rays do not undergo refraction at this side of the prism, and it is only the effect produced by the second surface which needs to be considered. Let *LD* and *LB* be two of these rays, striking the surface of exit at points *D* and *B*. Let *BC* be the direction taken by the ray *LB* when it leaves the prism, in the case when the prism is stationary. If a perpendicular is dropped from the point *D* onto the emerging ray, and starting from point *B* a line *BA* is drawn perpendicular to the incident ray, the light must travel from *A* to *D* in the same time as from *B* to *C*: this is the law determining the direction of the refracted wave *DC*. But as the prism is being carried along by the movement of the earth while the light is travelling from *A* to *D*,

point *D* is displaced, as a result of which the difference between the two paths travelled by the rays *LD* and *LB* is increased, which necessarily changes the angle of refraction. *FG* represents the surface of emergence; let *D′* be the point where, after the incident wave has reached *AB*, the ray *AD* strikes this surface and leaves the prism. Let *BC′* be the new direction of the refracted rays. The perpendicular *D′C′* will represent the direction of the emerging wave, which must satisfy the general condition that *AD′* is travelled by the light in the same time as *BC′*. But in order to establish the relative lengths of these two intervals, it is necessary to calculate the variation introduced by the movement of the prism in the velocity of the light waves travelling through it.

If this prism carried along with it all of the ether it contains, the whole of the medium acting as a vehicle for the waves would thus participate in the movement of the earth, and the velocity of the light waves would be equal to their velocity in a supposedly stationary medium, plus the velocity of the earth. But the case in point is more complicated; it is only a part of the medium which is carried along by our earth—namely, the proportion by which its density exceeds that of the surrounding ether. By analogy it would seem that when only a part of the medium is displaced, the velocity of propagation of waves can only be increased by the velocity of the centre of gravity of the system.

This principle is evident in the case where the moving part represents exactly half of the medium; for, relating the movement of the system to its centre of gravity, which is considered for a moment as fixed, its two halves are travelling away from one another at an equal velocity in opposite directions; it follows that the waves must be slowed down in one direction as much as they are accelerated in the other, and that in relation to the centre of gravity they thus travel only at their normal velocity of propagation; or, which amounts to the same thing, they share its movement. If the moving portion were one quarter, one eighth or one sixteenth, etc., of the medium, it could be just as easily shown that the velocity to be added to the velocity of wave propagation is one quarter, one eighth, one sixteenth, and so on, of that of the part in motion—that is to say, the exact velocity of the centre of gravity; and it is clear that a theorem which holds good in all these individual instances must be generally valid.

This being established, and the prismatic medium being in equilibrium of forces *(tension)* with the surrounding ether (I am supposing for the sake of simplicity that the experiment is conducted in vacuum), any delay the light undergoes when passing through the prism when it is stationary may be considered as a result solely of its greater density, which provides a means of determining the relative densities of the two media; for we know that their relationship must be the inverse of the squares of velocity of wave propagation. Let d and d' be the wavelengths of light in the surrounding ether and in the prism, $\triangle$ and $\triangle'$ the densities of these two media; this gives us the ratio:

$$d^2 : d'^2 :: \triangle' : \triangle;$$

whence we obtain:

$$\triangle' = \triangle \frac{d^2}{d'^2},$$

and consequently:

$$\triangle' - \triangle = \triangle\left(\frac{d^2-d'^2}{d'^2}\right).$$

This is the density of the mobile part of the prismatic medium. If the distance covered by the earth in the period of one light wave cycle is represented by t, the displacement of the centre of gravity of this medium during the same interval of time, which I am taking as a unit, or the velocity of this centre of gravity, will be:

$$t\left(\frac{d^2-d'^2}{d^2}\right).$$

Consequently the wavelength d'' within the prism being carried along by the earth will be equal to:

$$d' + t\left(\frac{d^2-d'^2}{d^2}\right).$$

By calculating, with the help of this expression, the interval AD' (Fig. 1.1) travelled by the ray AD before it leaves the prism, one may easily determine the direction of the refracted ray BC'. If this direction is compared to the direction of the ray BC obtained when the prism is stationary, the value for the sine of angle CBC', omitting all terms multiplied by the square or by greater powers of t because of the very small value of t, can be expressed as:

$$\frac{t}{d'} \sin i \cos i - \frac{t}{dd'} \sin i \sqrt{(d'^2 - d^2 \sin^2 i)},$$

where i represents the angle of incidence ABD.

Supposing that, from a point H somewhere on the ray BC, a line HH' is drawn parallel to the ecliptic and equal to the distance travelled by the earth during the time taken by the light to travel from B to H'; the optic axis of the telescope with which the object is being observed being directed along BH, the light must follow the direction BH' in order to arrive at H' at the same time as the cross-hair of the instrument which is being carried along with the movement of the earth: now, the line BH' coincides exactly with the direction BC' of the ray refracted by the prism, which is being carried along with the same movement; for the expression of the value of sine HBH' is also found to be:

$$\frac{t}{d'} \sin i \cos i - \frac{t}{dd'} \sin i \sqrt{(d'^2 - d^2 \sin^2 i)}.$$

Thus the telescope must be placed in the same direction as if the prism were stationary; whence it results that the movement of our earth cannot have any perceptible effect upon the apparent refraction, even supposing that it communicates only a very small part of its velocity to the ether. A very simple calculation confirms that the same is true of reflexion. Thus this hypothesis, which gives a satisfactory explanation for aberration, does not lead to any conclusion which contradicts the observed facts.

I shall conclude this letter with an application of the same theory to the experiment proposed by Boscovich, which consists in observing the phenomenon of aberration through instruments filled with water, or with some other fluid much more refractive than air, in order to ascertain whether the direction where a star is seen to lie varies as a result of the alteration in the course of the light introduced by the liquid. I shall first point out that it is unnecessary when trying to obtain this result to introduce the added complication of aberration; it can be obtained by observing a terrestrial object just as well as by observing a star. This, as it seems to me, is the simplest and most convenient way of conducting the experiment.

Having fixed relative to the instrument, or rather, to the microscope *FBDE*... [Fig. 1.2], the target *M* situated on the projection of the optic axis *CA*, this system should be placed at right angles to the ecliptic, and when the observation has been made in one direction, the whole system should be turned the other way about, and an observation made in the opposite direction. If the movement of the earth displaced the image of point *M* in relation to the cross-hair of the eye-piece, the image would appear now to the right, now to the left, of the cross-hair.

With the corpuscular theory it is clear, as Wilson has already pointed out, that the movement of the earth would not affect the appearance of this phenomenon. In fact, the movement of the earth means that a ray leaving *M* must take, in order to pass through the centre of the objective, a direction *MA′* such that the distance *AA′* is travelled by the earth in the same interval of time taken by the light to cover the distance *MA′*, or *MA* (on account of the low value of the earth relative to the velocity of

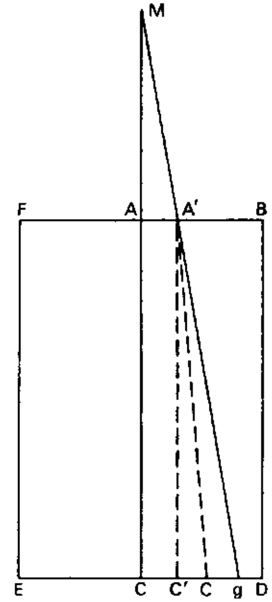

Fig. 1.2

light). When v represents the velocity of light in air, and t the velocity of the earth, we obtain:

$$MA : AA' :: v : t, \quad \text{or} \quad \frac{AA'}{MA} = \frac{t}{v};$$

which is the sine of the angle of incidence. When v' is the velocity of light in the denser medium contained in the instrument, the sine of the angle of refraction $C'A'G$ will be equal to t/v'; thus we obtain $C'G = A'C'(t/v')$; from which we derive the ratio:

$$C'G : A'C' :: t : v'.$$

Consequently the cross-hair C' of the eye-piece situated on the optic axis of the instrument will arrive at G at the same time as the light ray which has passed through the centre of the objective.

The theory of waves leads to the same results. I am supposing for the sake of simplicity that the microscope is in a vacuum. When d and d' represent the velocity of light in a vacuum and in the medium contained in the instrument, we find the value t/d for the sine of the angle of incidence AMA', and td'/d^2 for the sine of the angle of refraction $C'A'G$. Thus, independently of the displacement of the waves in the direction of the movement of the earth, $C'G = A'C'(td'/d^2)$. But the velocity with which these waves are carried along by the moving part of the medium in which they are propagated is equal to:

$$t\left(\frac{d^2-d'^2}{d^2}\right);$$

thus their total displacement Gg, during the time they take to travel through the microscope, equals:

$$\frac{A'C'}{d'}\, t\left(\frac{d^2-d'^2}{d^2}\right);$$

thus:

$$C'g = A'C'\cdot t\left(\frac{d'}{d^2}+\frac{d^2-d'^2}{d'd^2}\right) = A'C'\cdot t\left(\frac{d^2}{d'd^2}\right) = A'C'\cdot\frac{t}{d'}.$$

Thus we obtain the ratio $C'g : A'C' :: t : d'$; consequently the image of the point M will arrive at g at the same time as the cross-hair of the micrometer. Thus the appearance of the phenomenon must always remain the same, whatever the direction in which the instrument is turned. Although this experiment has not yet been performed, I do not doubt but that it would confirm this conclusion, which is deducible equally from the corpuscular theory and from the wave theory.

Additional note to the letter

(*Annales de chimie et de physique*, Vol. IX, p. 286, November 1818)

In calculating the refraction of light in a prism being carried along by the movement of the earth, I supposed, in order to

simplify the reasoning, that the difference between the velocity of light in the prism and in the ether surrounding it was solely a result of the difference in density, elasticity being the same in the two cases; but it is very possible that the two media differ in elasticity as in density. It would even be conceivable that the elasticity of a solid body might vary according to the direction from which it was considered; and it is very probably this which gives rise to double refraction, as Dr. Young has observed. But whatever hypothesis is formed concerning the causes of the slowing of light when it passes through transparent bodies, it is always possible, in order to resolve the problem which was set me, to substitute in thought for the real medium of the prism, an elastic fluid in equilibrium of forces with the surrounding ether, and having a density such that the velocity of light is exactly the same in this fluid and in the prism, when they are supposed stationary; this equality must still remain when the two media are being carried along by the movement of the earth: these, then, are the bases upon which my calculations rest.

2. ON THE ABERRATION OF LIGHT*

G. G. Stokes

The general explanation of the phenomenon of aberration is so simple, and the coincidence of the value of the velocity of light thence deduced with that derived from the observations of the eclipses of Jupiter's satellites so remarkable, as to leave no doubt on the mind as to the truth of the explanation. But when we examine the cause of the phenomenon more closely, it is far from being so simple as it appears at first sight. On the theory of emissions, indeed, there is little difficulty; and it would seem that the more particular explanation of the cause of aberration usually given, which depends on the consideration of the motion of a telescope as light passes from its object-glass to its cross wires, has reference especially to this theory; for it does not apply to the theory of undulations, unless we make the rather startling hypothesis that the luminiferous ether passes freely through the sides of the telescope and through the earth itself. The undulatory theory of light, however, explains so simply and so beautifully the most complicated phenomena, that we are naturally led to regard aberration as a phenomenon unexplained by it, but not incompatible with it.

* Reprinted from G. G. Stokes, *Math. and Phys. Papers*, **1**, 134–40. The first part is unchanged from original (1845) publication in *Phil. Mag.*, **27**, 9; The "Additional Note" was substituted in 1880 as an unprovement of the 1845 argument to the same end.

The object of the present communication is to attempt an explanation of the cause of aberration which shall be in accordance with the theory of undulations. I shall suppose that the earth and the planets carry a portion of the ether along with them so that the ether close to their surfaces is at rest relatively to those surfaces, while its velocity alters as we recede from the surface, till, at no great distance, it is at rest in space. According to the undulatory theory, the direction in which a heavenly body is seen is normal to the fronts of the waves which have emanated from it, and have reached the neighbourhood of the observer, the ether near him being supposed to be at rest relatively to him. If the ether in space were at rest, the front of a wave of light at any instant being given, its front at any future time could be found by the method explained in Airy's tracts. If the ether were in motion, and the velocity of propagation of light were infinitely small, the wave's front would be displaced as a surface of particles of the ether. Neither of these suppositions is, however, true, for the ether moves while light is propagated through it. In the following investigation I suppose that the displacements of a wave's front in an elementary portion of time due to the two causes just considered take place independently.

Let u, v, w be the resolved parts along the rectangular axes of x, y, z, of the velocity of the particle of ether whose co-ordinates are x, y, z, and let V be the velocity of light supposing the ether at rest. In consequence of the distance of the heavenly bodies, it will be quite unnecessary to consider any waves except those which are plane, except in so far as they are distorted by the motion of the ether. Let the axis of z be taken in, or nearly in the direction of propagation of the wave considered, so that the equation of a wave's front at any time will be

$$z = C + Vt + \zeta, \tag{1}$$

C being a constant, t the time, and ζ a small quantity, a function of x, y and t. Since u, v, w and ζ are of the order of the aberration, their squares and products may be neglected.

Denoting by α, β, γ the angles which the normal to the wave's front at the point (x, y, z) makes with the axes, we have, to the first order of approximation,

$$\cos\alpha = -\frac{d\zeta}{dx}, \quad \cos\beta = -\frac{d\zeta}{dy}, \quad \cos\gamma = 1; \tag{2}$$

and if we take a length $V\,dt$ along this normal, the co-ordinates of its extremity will be

$$x-\frac{d\zeta}{dx}V\,dt, \quad y-\frac{d\zeta}{dy}V\,dt, \quad z+V\,dt.$$

If the ether were at rest, the locus of these extremities would be the wave's front at the time $t+dt$, but since it is in motion, the co-ordinates of those extremities must be further increased by $u\,dt$, $v\,dt$, $w\,dt$. Denoting then by x', y', z' the co-ordinates of the point of the wave's front at the time $t+dt$ which corresponds to the point (x, y, z) at the time t, we have

$$x' = x+\left(u-V\frac{d\zeta}{dx}\right)dt, \quad y' = y+\left(v-V\frac{d\zeta}{dy}\right)dt,$$
$$z' = z+(w+V)\,dt;$$

and eliminating x, y and z from these equations and (1), and denoting ζ by $f(x, y, t)$, we have for the equation to the wave's front at the time $t+dt$,

$$z'-(w+V)\,dt = C+Vt$$
$$+f\left\{x'-\left(u+\frac{d\zeta}{dx}\right)dt, \quad y'-\left(v+\frac{d\zeta}{dy}\right)dt, t\right\},$$

or, expanding, neglecting dt^2 and the square of the aberration, and suppressing the accents of x, y and z,

$$z = C+Vt+\zeta+(w+V)\,dt. \tag{3}$$

But from the definition of ζ it follows that the equation to the wave's front at the time $t+dt$ will be got from (1) by putting $t+dt$ for t, and we have therefore for this equation

$$z = C+Vt+\zeta+\left(V+\frac{d\zeta}{dt}\right)dt. \tag{4}$$

Comparing the identical equations (3) and (4), we have

$$\frac{d\zeta}{dt} = w.$$

This equation gives $\zeta = \int w\,dt$; but in the small term ζ we may replace $\int w\,dt$ by $\int w\,dz \div V$: this comes to taking the approximate value of z given by the equation $z = C+Vt$ instead of t for the parameter of the system of surfaces formed by the wave's front in its successive positions. Hence equation (1) becomes

$$z = C+Vt+\frac{1}{V}\int w\,dz.$$

Combining the value of ζ just found with equations (2), we get, to a first approximation,

$$\alpha-\frac{\pi}{2} = \frac{1}{V}\int\frac{dw}{dx}dz, \quad \beta-\frac{\pi}{2} = \frac{1}{V}\int\frac{dw}{dy}dz, \tag{5}$$

equations which might very easily be proved directly in a more geometrical manner.

If random values are assigned to u, v and w, the law of aberration resulting from these equations will be a complicated one; but if u, v and w are such that $u\,dx+v\,dy+w\,dz$ is an exact differential, we have,

$$\frac{dw}{dx} = \frac{du}{dz}, \quad \frac{dw}{dy} = \frac{dv}{dz};$$

whence, denoting by the suffixes 1, 2 the values of the variables belonging to the first and second limits respectively, we obtain

$$\alpha_2-\alpha_1 = \frac{u_2-u_1}{V}, \quad \beta_2-\beta_1 = \frac{v_2-v_1}{V}. \qquad (6)$$

If the motion of the ether be such that $u\,dx+v\,dy+w\,dz$ is an exact differential for one system of rectangular axes, it is easy to prove, by the transformation of co-ordinates, that it is an exact differential for any other system. Hence the formulæ (6) will hold good, not merely for light propagated in the direction first considered, but for light propagated in any direction, the direction of propagation being taken in each case for the axis of z. If we assume that $u\,dx+v\,dy+w\,dz$ is an exact differential for that part of the motion of the ether which is due to the motion of translation of the earth and planets, it does not therefore follow that the same is true for that part which depends on their motions of rotation. Moreover, the diurnal aberration is too small to the detected by observation, or at least to be measured with any accuracy, and I shall therefore neglect it.

It is not difficult to shew that the formulæ (6) lead to the known law of aberration. In applying them to the case of a star, if we begin the integrations in equations (5) at a point situated at such a distance from the earth that the motion of the ether, and consequently the resulting change in the direction of the light, is insensible, we shall have $u_1 = 0$, $v_1 = 0$; and if, moreover, we take the plane xz to pass through the direction of the earth's motion, we shall have

$$v_2 = 0, \quad \beta_2-\beta_1 = 0,$$

and

$$\alpha_2-\alpha_1 = \frac{u_2}{V};$$

that is, the star will appear displaced towards the direction in which the earth is moving, through an angle equal to the ratio of

the velocity of the earth to that of light, multiplied by the sine of the angle between the direction of the earth's motion and the line joining the earth and the star.

Additional Note

[In what precedes *waves* of light are alone considered, and the course of a *ray* is not investigated, the investigation not being required. There follows in the original paper an investigation having for object to shew that in the case of a body like the moon or a planet which is itself in motion, the effect of the distortion of the waves in the neighbourhood of the body in altering the apparent place of the body as determined by observation is insensible. For this, the orthogonal trajectory of the wave in its successive positions from the body to the observer is considered, a trajectory which in its main part will be a straight line, from which it will not differ except in the immediate neighbourhood of the body and of the earth, where the ether is distorted by their respective motions. The perpendicular distance of the further extremity of the trajectory from the prolongation of the straight line which it forms in the intervening quiescent ether is shewn to subtend at the earth an angle which, though not actually 0, is so small that it may be disregarded.

The orthogonal trajectory of a wave in its successive positions does not however represent the course of a ray, as it would do if the ether were at rest. Some remarks made by Professor Challis in the course of discussion suggested to me the examination of the path of a ray, which in the case in which $u\,dx+v\,dy+w\,dz$ is an exact differential proved to be a straight line, a result which I had not foreseen when I wrote the above paper, which I may mention was read before the Cambridge Philosophical Society on the 18th of May, 1845 (see *Philosophical Magazine*, vol. xxix, p. 62). The rectilinearity of the path of a ray in this case, though not expressly mentioned by Professor Challis, is virtually contained in what he wrote. The problem is rather simplified by introducing the consideration of rays, and may be treated from the beginning in the following manner.

The notation in other respects being as before, let α', β' be the small angles by which the direction of the wave-normal at the point (x, y, z) deviates from that of Oz towards Ox, Oy, respectively, so that α', β' are the complements of α, β, and let $\alpha_{,}$, $\beta_{,}$ be the inclinations to Oz of the course of a ray at the same point. By compounding the velocity of propagation through the ether with the velocity of the ether we easily see that

$$\alpha_{,} = \alpha' + \frac{u}{V}, \quad \beta_{,} = \beta' + \frac{v}{V}.$$

Let us now trace the changes of $\alpha_{,}$, $\beta_{,}$ during the time dt. These depend first on the changes of α', β', and secondly on those of u, v.

As regards the change in the direction of the wave-normal, we notice that the seat of a small element of the wave in its successive positions is in a succession of planes of particles nearly parallel to the plane of x, y. Consequently the direction of the element of the wave will be altered during the time dt by the motion of the ether as much as a plane of particles of the ether parallel to the plane of the wave, or, which is the same to the order of small quantities retained, parallel to the plane xy. Now if we consider a particle of ether at the time t having for co-ordinates x, y, z, another at a distance dx parallel to the axis of x, and a third at a distance dy parallel to the axis of y, we see that the displacements of these three particles parallel to the axis of z during the time dt will be

$$w\,dt, \quad \left(w + \frac{dw}{dx}\,dx\right) dt, \quad \left(w + \frac{dw}{dy}\,dy\right) dt;$$

and dividing the relative displacements by the relation distances, we have $dw/dx \cdot dt$, $dw/dy \cdot dt$ for the small angles by which the normal is displaced, in the planes of xz, yz, from the axes x, y, so that

$$d\alpha' = -\frac{dw}{dx}\,dt, \quad d\beta' = -\frac{dw}{dy}\,dt.$$

We have seen already that the changes of u, v are $du/dz. V\,dt$, $dv/dz. V\,dt$, so that

$$d\alpha_{,} = \left(\frac{du}{dz} - \frac{dw}{dx}\right) dt, \quad d\beta_{,} = \left(\frac{dv}{dz} - \frac{dw}{dy}\right) dt.$$

Hence, provided the motion of the ether be such that

$$u\,dx + v\,dy + w\,dz$$

is an exact differential, the change of direction of a ray as it travels along is *nil*, and therefore the course of a ray is a straight line notwithstanding the motion of the ether. The rectilinearity of propagation of a ray of light, which *a priori* would seem very likely to be interfered with by the motion of the ether produced by the earth or heavenly body moving through it, is the tacit assumption made in the explanation of aberration given in treatises of Astronomy, and provided that be accounted for the rest follows as usual.† It follows further that the angle subtended at the earth by the perpendicular distance of the point where a ray leaves a heavenly body from the straight line prolonged which represents its course through the intervening quiescent ether, is not merely too small to be observed, but actually *nil*.]

† To make this explanation *quite* complete, we should properly, as Professor Challis remarks, consider the light coming from the wires of the observing telescope, in company with the light from the heavenly body.

3. ON THE RELATIVE MOTION OF THE EARTH AND THE LUMINIFEROUS ÆTHER[*]

A. A. MICHELSON *and* E. W. MORLEY[†]

THE discovery of the aberration of light was soon followed by an explanation according to the emission theory. The effect was attributed to a simple composition of the velocity of light with the velocity of the earth in its orbit. The difficulties in this apparently sufficient explanation were overlooked until after an explanation on the undulatory theory of light was proposed. This new explanation was at first almost as simple as the former. But it failed to account for the fact proved by experiment that the aberration was unchanged when observations were made with a telescope filled with water. For if the tangent of the angle of aberration is the ratio of the velocity of the earth to the velocity of light, then, since the latter velocity in water is three-fourths its velocity in a vacuum, the aberration observed with a water telescope should be four-thirds of its true value.[‡]

[*] *Phil. Mag.* (5) **24**, 449 (1887). This version of Michelson and Morley's article is essentially equivalent to the American version published simultaneously in *Am. J. Sci.* (3), **34**, 333 (1887).

[†] Communicated by the Authors.

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[‡] It may be noticed that most writers admit the sufficiency of the explanation according to the emission theory of light; while in fact the difficulty is even greater than according to the undulatory theory. For on the emission

On the undulatory theory, according to Fresnel, first, the æther is supposed to be at rest, except in the interior of transparent media, in which, secondly, it is supposed to move with a velocity less than the velocity of the medium in the ratio $(n^2-1)/n^2$, where n is the index of refraction. These two hypotheses give a complete and satisfactory explanation of aberration. The second hypothesis, notwithstanding its seeming improbability, must be considered as fully proved, first, by the celebrated experiment of Fizeau,[*] and secondly, by the ample confirmation of our own work.[†] The experimental trial of the first hypothesis forms the subject of the present paper.

If the earth were a transparent body, it might perhaps be conceded, in view of the experiments just cited, that the intermolecular æther was at rest in space, not withstanding the motion of the earth in its orbit; but we have no right to extend the conclusion from these experiments to opaque bodies. But there can hardly be any question that the æther can and does pass through metals. Lorentz cites the illustration of a metallic barometer tube. When the tube is inclined, the æther in the space above the mercury is certainly forced out, for it is incompressible.[‡] But again we have no right to assume that it makes its escape with perfect freedom, and if there be any resistance, however slight, we certainly could not assume an opaque body such as the whole earth to offer free passage through its entire mass. But as Lorentz aptly remarks:

theory the velocity of light must be greater in the water telescope, and therefore the angle of aberration should be less; hence, in order to reduce it to its true value, we must make the absurd hypothesis that the motion of the water in the telescope carries the ray of light in the opposite direction!

[*] *Comptes Rendus*, xxxiii. p. 349 (1851); *Pogg. Ann.* Ergänzungsband, iii. p. 457 (1853); *Ann. Chim. Phys.* [3], lvii. p. 385 (1859).

[†] "Influence of Motion of the Medium on the Velocity of Light." *Am. J. Sci.* [3], xxxi. p. 377 (1886).

[‡] It may be objected that it may escape by the space between the mercury and the walls; but this could be prevented by amalgamating the latter.

"Quoi qu'il en soit, on fera bien, à mon avis, de ne pas se laisser guider, dans une question aussi importante, par des considérations sur le degré de probabilité ou de simplicité de l'une ou de l'autre hypothèse, mais de s'addresser a l'expérience pour appendre à connaître l'état, de repos ou de mouvement, dans lequel se trouve l'éther à la surface terrestre."*

In April, 1881, a method was proposed and carried out for testing the question experimentally.†

In deducing the formula for the quantity to be measured, the effect of the motion of the earth through the æther on the path of the ray at right angles to this motion was overlooked‡. The discussion of this oversight and of the entire experiment forms the subject of a very searching analysis by H. A. Lorentz§, who finds that this effect can by no means be disregarded. In consequence, the quantity to be measured had in fact but half the value supposed, and as it was already barely beyond the limits of errors of experiment, the conclusion drawn from the result of the experiment might well be questioned; since, however, the main portion of the theory remains unquestioned, it was decided to repeat the experiment with such modifications as would insure a theoretical result much too large to be masked by experimental errors. The theory of the method may be briefly stated as follows:—

Let *sa*, fig. 3.1, be a ray of light which is partly reflected in *ab*, and partly transmitted in *ac*, being returned by the mirrors *b* and *c* along *ba* and *ca*. *ba* is partly transmitted along *ad*, and *ca* is partly reflected along *ad*. If then the paths *ab* and *ac* are equal, the two rays interfere along *ad*. Suppose now, the æther being at rest, that the whole apparatus moves in the direction *sc*, with the velocity

* *Archives Néerlandaises*, xxi. 2me livr. *Phil. Mag.* [5], xiii. p. 236.

† "The Relative Motion of the Earth and the Luminiferous Aether," by Albert A. Michelson. *Am. J. Sci.* [3], xxii. p. 120.

‡ It may be mentioned here that the error was pointed out to the author of the former paper by M. A. Potier, of Paris, in the winter of 1881.

§ "De l'Influence du Mouvement de la Terre sur les Phen. Lum." *Archives Néerlandaises*, xxi. 2me livr. (1886).

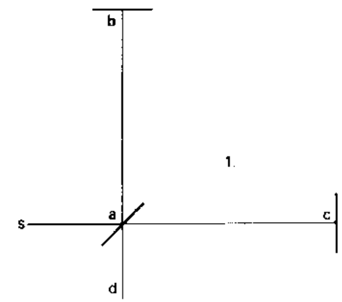


FIG. 3.1

of the earth in its orbit, the directions and distances traversed by the rays will be altered thus:—The ray *sa* is reflected along *ab*, fig. 3.2; the angle $bab_{,}$ being equal to the aberration $= \alpha$, is returned along $ba_{,}$, ($aba_{,} = 2\alpha$), and goes to the focus of the telescope, whose direction is unaltered. The transmitted ray goes along *ac*, is returned along $ca_{,}$, and is reflected at $a_{,}$, making $ca_{,}e$ equal $90-\alpha$, and therefore still coinciding with the first ray. It may be

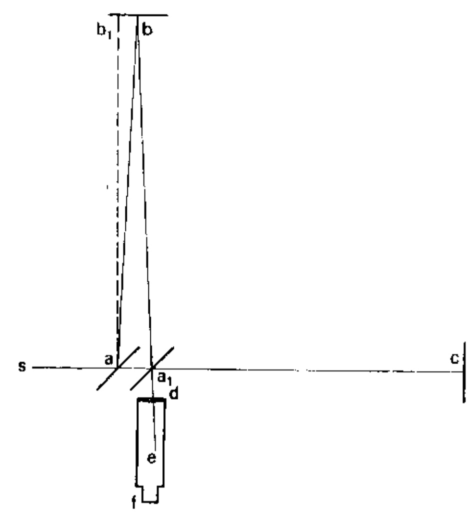


FIG. 3.2

remarked that the rays $ba_{\prime}$ and $ca_{\prime}$ do not now meet exactly in the same point $a_{\prime}$, though the difference is of the second order; this does not affect the validity of the reasoning. Let it now be required to find the difference in the two paths $aba_{\prime}$ and $aca_{\prime}$.

Let V = velocity of light.
v = velocity of the earth in its orbit.
D = distance *ab* or *ac*, Fig. 3.1.
T = time light occupies to pass from *a* to *c*.
$T_{\prime}$ = time light occupies to return from *c* to $a_{\prime}$ (Fig. 3.2).

Then

$$T = \frac{D}{V-v} \quad T_{\prime} = \frac{D}{V+v}.$$

The whole time of going and coming is

$$T+T_{\prime} = 2D\frac{V}{V^2-v^2},$$

and the distance travelled in this time is

$$2D\frac{V^2}{V^2-v^2} = 2D\left(1+\frac{v^2}{V^2}\right),$$

neglecting terms of the fourth order. The length of the other path is evidently

$$2D\sqrt{\left(1+\frac{v^2}{V^2}\right)},$$

or to the same degree of accuracy,

$$2D\left(1+\frac{v^2}{2V^2}\right).$$

The difference is therefore $D(v^2/V^2)$. If now the whole apparatus be turned through 90°, the difference will be in the opposite direction, hence the displacement of the interference-fringes should be $2D(v^2/V^2)$. Considering only the velocity of the earth in its orbit, this would be $2D\times10^{-8}$. If, as was the case in the first experiment,

$D = 2\times10^6$ waves of yellow light, the displacement to be expected would be 0·04 of the distance between the interference-fringes.

In the first experiment, one of the principal difficulties encountered was that of revolving the apparatus without producing distortion; and another was its extreme sensitiveness to vibration. This was so great that it was impossible to see the interference-fringes except at brief intervals when working in the city, even at two o'clock in the morning. Finally, as remarked before, the quantity to be observed, namely, a displacement of something less than a twentieth of the distance between the interference-fringes, may have been too small to be detected when masked by experimental errors.

The first-named difficulties were entirely overcome by mounting the apparatus on a massive stone floating on mercury; and the second by increasing, by repeated reflexion, the path of the light to about ten times its former value.

The apparatus is represented in perspective in fig. 3.3, in plan in fig. 3.4, and in vertical section in fig. 3.5. The stone *a* (fig. 3.5) is about 1·5 metre square and 0·3 metre thick. It rests on an annular wooden float *bb*, 1·5 metre outside diameter, 0·7 metre inside diameter, and 0·25 metre thick. The float rests on mercury contained in the cast-iron trough *cc*, 1·5 centimetre thick, and of such dimensions as to leave a clearance of about one centimetre around the float. A pin *d*, guided by arms *g g g g*, fits into a socket *e* attached to the float. The pin may be pushed into the socket or be withdrawn, by a lever pivoted at *f*. This pin keeps the float concentric with the trough, but does not bear any part of the weight of the stone. The annular iron trough rests on a bed of cement on a low brick pier built in the form of a hollow octagon.

At each corner of the stone were placed four mirrors *dd ee*, fig. 3.4. Near the centre of the stone was a plane parallel glass *b*. These were so disposed that light from an argand burner *a*, passing through a lens, fell on *b* so as to be in part reflected to $d_{\prime}$; the two pencils followed the paths indicated in the figure, *b d e d b f* and

$b\ d_1\ e_1\ d_1\ b\ f$ respectively, and were observed by the telescope *f*. Both *f* and *a* revolved with the stone. The mirrors were of speculum metal carefully worked to optically plane surfaces five centimetres in diameter, and the glasses *b* and *c* were plane parallel of the same

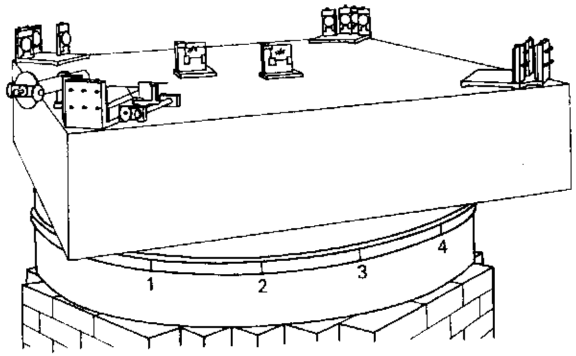


FIG. 3.3

thickness, 1·25 centimetre; their surfaces measured 5·0 by 7·5 centimetres. The second of these was placed in the path of one of the pencils to compensate for the passage of the other through the same thickness of glass. The whole of the optical portion of the apparatus was kept covered with a wooden cover to prevent air-currents and rapid changes of temperature.

The adjustment was effected as follows:—The mirrors having been adjusted by screws in the castings which held the mirrors, against which they were pressed by springs, till light from both pencils could be seen in the telescope, the lengths of the two paths measured by a light wooden rod reaching diagonally from mirror to mirror, the distance being read from a small steel scale to tenths of millimetres. The difference in the lengths of the two paths was then annulled by moving the mirror e_1. This mirror had three adjustments: it had an adjustment in altitude and one in azimuth, like all the other mirrors, but finer; it also had an adjustment in the direction of the incident ray, sliding forward or back-

ward, but keeping very accurately parallel to its former plane. The three adjustments of this mirror could be made with the wooden cover in position.

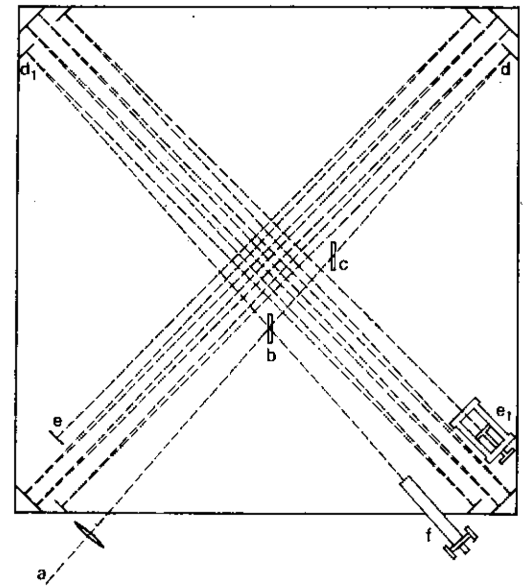


FIG. 3.4

The paths being now approximately equal, the two images of the source of light or of some well-defined object placed in front of the condensing lens, were made to coincide, the telescope was now adjusted for distinct vision of the expected interference-bands, and sodium light was substituted for white light, when the interference-bands appeared. These were now made as clear as possible by adjusting the mirror e_1; then white light was restored, the screw altering the length of path was very slowly moved (one turn of a screw of one hundred threads to the inch altering the path nearly 1000 wave-lengths) till the coloured interference-fringes reappeared in white light. These were now given a convenient width and position, and the apparatus was ready for observation.

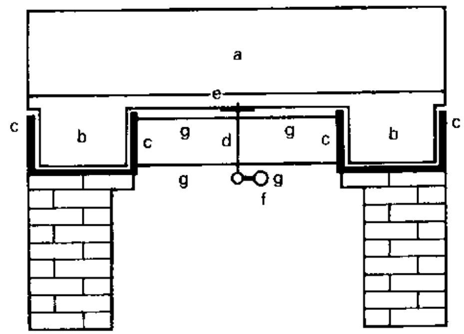


FIG. 3.5

The observations were conducted as follows:—Around the cast-iron trough were sixteen equidistant marks. The apparatus was revolved very slowly (one turn in six minutes) and after a few minutes the cross wire of the micrometer was set on the clearest of the interference-fringes at the instant of passing one of the marks. The motion was so slow that this could be done readily and accurately. The reading of the screw-head on the micrometer was noted, and a very slight and gradual impulse was given to keep up the motion of the stone; on passing the second mark, the same process was repeated, and this was continued till the apparatus had completed six revolutions. It was found that by keeping the apparatus in slow uniform motion, the results were much more uniform and consistent than when the stone was brought to rest for every observation; for the effects of strains could be noted for at least half a minute after the stone came to rest, and during this time effects of change of temperature came into action.

The following tables give the means of the six readings; the first, for observations made near noon, the second, those near six o'clock in the evening. The readings are divisions of the screw-heads.

The width of the fringes varied from 40 to 60 divisions, the mean value being near 50, so that one division means 0·02 wave-length. The rotation in the observations at noon was contrary to, and in the evening observations, in the same direction as, that of the hands of a watch.

NOON OBSERVATIONS

	16.	1.	2.	3.	4.	5.	6.	7.	8.
July 8	44·7	44·0	43·5	39·7	35·2	34·7	34·3	32·5	28·2
July 9	57·4	57·3	58·2	59·2	58·7	60·2	60·8	62·0	61·5
July 11	27·3	23·5	22·0	19·3	19·2	19·3	18·7	18·8	16·2
Mean	43·1	41·6	41·2	39·4	37·7	38·1	37·9	37·8	35·3
Mean in w.l.	·862	·832	·824	·788	·754	·762	·758	·756	·700
	·706	·692	·686	·688	·688	·678	·672	·628	·616
Final mean	·784	·762	·755	·738	·721	·720	·715	·692	·661

	9.	10.	11.	12.	13.	14.	15.	16.
July 8	26·2	23·8	23·2	20·3	18·7	17·5	16·8	13·7
July 9	63·3	65·8	67·3	69·7	70·7	73·0	70·2	72·2
July 11	14·3	13·3	12·8	13·3	12·3	10·2	7·3	6·5
Mean	34·6	34·3	34·4	34·4	33·9	33·6	31·4	30·8
Mean in w.l.	·692	·686	·688	·688	·678	·672	·628	·616
Final mean	...	...	...	...	...	...	...	...

The results of the observations are expressed graphically in fig. 3.6. The upper is the curve for the observations at noon, and the lower that for the evening observations. The dotted curves represent *one eighth* of the theoretical displacements. It seems fair to conclude from the figure that if there is any displacement due to the relative motion of the earth and the luminiferous æther, this cannot be much greater than 0·01 of the distance between the fringes.

P.M. OBSERVATIONS

	16.	1.	2.	3.	4.	5.	6.	7.	8.
July 8	61·2	63·3	63·3	68·2	67·7	69·3	70·3	69·8	69·0
July 9	26·0	26·0	28·2	29·2	31·5	32·0	31·3	31·7	33·0
July 12	66·8	66·5	66·0	64·3	62·2	61·0	61·3	58·7	58·4
Mean	51·3	51·9	52·5	53·9	53·9	54·1	54·3	53·7	53·4
Mean in w.l.	1·026	1·038	1·050	1·078	1·076	1·082	1·086	1·074	1·068
	1·068	1·086	1·076	1·084	1·100	1·136	1·144	1·154	1·172
Final mean	1·047	1·062	1·063	1·081	1·088	1·109	1·115	1·114	1·120

	9.	10.	11.	12.	13.	14.	15.	16.
July 8	71·3	71·3	70·5	71·2	71·2	70·5	72·5	75·7
July 9	35·8	36·5	37·3	38·8	41·0	42·7	43·7	44·0
July12	55·7	53·7	54·7	55·0	58·2	58·5	57·0	56·0
Mean	54·3	53·8	54·2	55·0	56·8	57·2	57·7	58·6
Mean in w.l.	1·086	1·076	1·084	1·100	1·136	1·144	1·154	1·172
Final mean	...	...	...	...	...	...	...	...

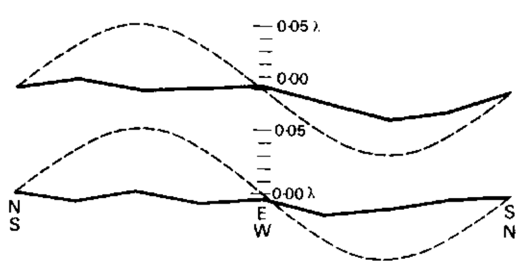


FIG. 3.6

Considering the motion of the earth in its orbit only, this displacement should be

$$2D\frac{v^2}{V^2} = 2D\times 10^{-8}.$$

The distance D was about eleven metres, or 2×10^7 wavelengths of yellow light; hence the displacement to be expected was 0·4 fringe. The actual displacement was certainly less than the twentieth part of this, and probably less than the fortieth part. But since the displacement is proportional to the square of the velocity, the relative velocity of the earth and the æther is probably less than one sixth the earth's orbital velocity, and certainly less than one fourth.

In what precedes, only the orbital motion of the earth is considered. If this is combined with the motion of the solar system, concerning which but little is known with certainty, the result would have to be modified; and it is just possible that the resultant velocity at the time of the observations was small, though the chances are much against it. The experiment will therefore be repeated at intervals of three months, and thus all uncertainty will be avoided.

It appears from all that precedes reasonably certain that if there be any relative motion between the earth and the luminiferous æther, it must be small; quite small enough entirely to refute Fresnel's explanation of aberration. Stokes has given a theory of aberration which assumes the aether at the earth's surface to be at rest with regard to the latter, and only requires in addition that the relative velocity have a potential; but Lorentz shows that these conditions are incompatible. Lorentz then proposes a modification which combines some ideas of Stokes and Fresnel, and assumes the existence of a potential, together with Fresnel's coefficient. If now it were legitimate to conclude from the present work that the æther is at rest with regard to the earth's surface, according to Lorentz there could not be a velocity potential, and his own theory also fails.

Supplement

It is obvious from what has gone before that it would be hopeless to attempt to solve the question of the motion of the solar system by observations of optical phenomena *at the surface of the earth.* But it is not impossible that at even moderate distances above the level of the sea, at the top of an isolated mountain-peak, for instance, the relative motion might be perceptible in an apparatus like that used in these experiments. Perhaps if the experiment should ever be tried under these circumstances, the cover should be of glass, or should be removed.

It may be worth while to notice another method for multiplying the square of the aberration sufficiently to bring it within the range of observation which has presented itself during the preparation of this paper. This is founded on the fact that reflexion from surfaces in motion varies from the ordinary laws of reflexion.

Let *ab* (fig. 1, p. 158) be a plane wave falling on the mirror *mn* at an incidence of 45°. If the mirror is at rest, the wave-front after reflexion will be *ac*.

Now suppose the mirror to move in a direction which makes an angle α with its normal, with a velocity ω. Let V be the velocity of light in the æther, supposed stationary, and let *cd* be the increase in the distance the light has to travel to reach *d*. In this time the mirror will have moved a distance

$$\frac{cd}{\sqrt{(2\cos\alpha)}}.$$

We have

$$\frac{cd}{ad} = \frac{\omega\sqrt{(2\cos\alpha)}}{V},$$

which put $= r$, and

$$\frac{ac}{ad} = 1-r.$$

In order to find the new wave-front, draw the arc *fg* with *b* as a centre and *ad* as radius; the tangent to this arc from *d* will be the new wave-front, and the normal to the tangent from *b* will be the new direction. This will differ from the direction *ba* by θ, which it is required to find. From the equality of the triangles *adb* and *edb* it follows that $\theta = 2\phi$, $ab = ac$,

$$\tan adb = \tan\left(45^\circ - \frac{\theta}{2}\right) = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \frac{ac}{ad} = 1-r,$$

or, neglecting terms of the order r^3,

$$\theta = r + \frac{r^2}{2} = \frac{\sqrt{(2\omega\cos\alpha)}}{V} + \frac{\omega^2}{V^2}\cos^2\alpha.$$

Now let the light fall on a parallel mirror facing the first, we should then have

$$\theta_{,} = \frac{-\sqrt{(2\omega\cos\alpha)}}{V} + \frac{\omega^2}{V^2}\cos_1\alpha,$$

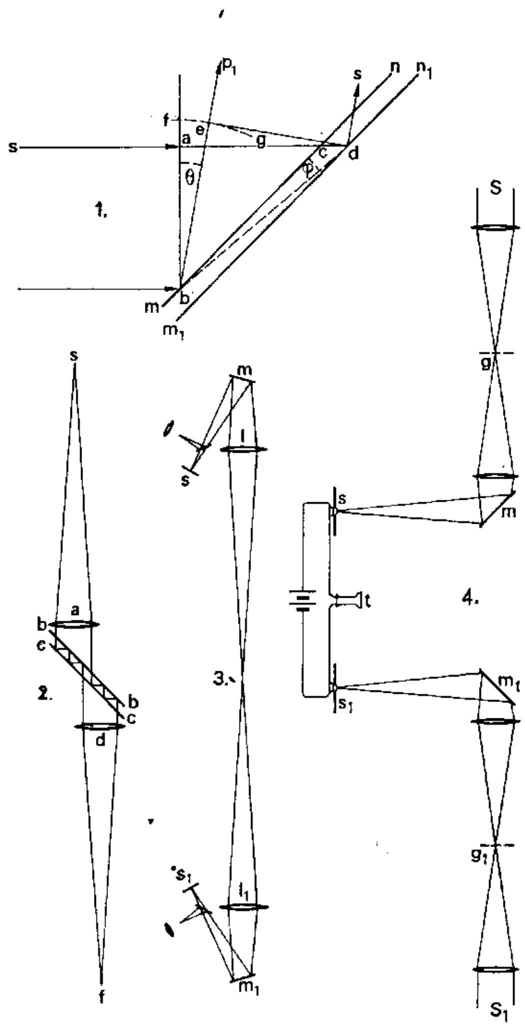
and the total deviation would be

$$\theta + \theta_{,} = 2\varrho^2\cos^2\alpha,$$

where ϱ is the angle of aberration, if only the orbital motion of the earth is considered. The maximum displacement obtained by revolving the whole apparatus through 90° would be

$$\triangle = 2\varrho^2 = 0{\cdot}004''.$$

With fifty such couples the displacement would be 0·2″. But astronomical observations in circumstances far less favourable than those in which these may be taken have been made to hundredths of a second; so that this new method bids fair to be at least as sensitive as the former.



The arrangement of apparatus might be as in fig. 2; *s*, in the focus of the lens *a*, is a slit. $bb_,$ $cc_,$ are two glass mirrors optically plane, and so silvered as to allow say one twentieth of the light to pass through, and reflecting say ninety per cent. The intensity of the light falling on the observing telescope *df* would be about one millionth of the original intensity, so that if sunlight or the electric arc were used it could still be readily seen. The mirrors $bb_,$ and $cc_,$ would differ from parallelism sufficiently to separate the successive images. Finally, the apparatus need not be mounted so as to revolve, as the earth's rotation would be sufficient.

If it were possible to measure with sufficient accuracy the velocity of light without returning the ray to its starting point, the problem of measuring the first power of the relative velocity of the earth with respect to the æther would be solved. This may not be as hopeless as might appear at first sight, since the difficulties are entirely mechanical and may possibly be surmounted in the course of time.

For example, suppose *m* and $m_,$ (fig. 3) two mirrors revolving with equal velocity in opposite directions. It is evident that light from *s* will form a stationary image at $s_,$ and similarly light from $s_,$ will form a stationary image at *s*. If now the velocity of the mirrors be increased sufficiently, their phases still being exactly the same, both images will be deflected from *s* and $s_,$ in inverse proportion to the velocities of light in the two directions; or, if the two deflections are made equal, and the difference of phase of the mirrors be simultaneously measured, this will evidently be proportional to the difference of velocity in the two directions. The only real difficulty lies in this measurement. The following is perhaps a possible solution.

$gg_,$ (fig. 4) are two gratings on which sunlight is concentrated. These are placed so that after falling on the revolving mirrors *m* and $m_,$, the light forms images of the gratings at *s* and $s_,$, two very sensitive selenium cells in circuit with a battery and telephone. If everything be symmetrical, the sound in the telephone will be a maximum. If now one of the slits *s* be displaced through half the distance between the image of the grating bars, there will be silence. Suppose now that the two deflections having been made exactly equal, the slit is adjusted for silence. Then if the experiment be repeated when the earth's rotation has turned the whole apparatus through 180°, and the deflections are again made equal, there will no longer be silence, and the angular distance through which *s* must be moved to restore silence will measure the required difference in phase.

There remain three other methods, all astronomical, for attacking the problem of the motion of the solar system through space.

1. The telescopic observation of the proper motions of the stars. This has given us a highly probably determination of the direction of this motion, but only a guess as to its amount.

2. The spectroscopic observation of the motion of stars in the line of sight. This could furnish data for the relative motions only, though it seems likely that by the immense improvements in the photography of stellar spectra, the information thus obtained will be far more accurate than any other.

3. Finally there remains the determination of the velocity of light by observations of the eclipses of Jupiter's satellites. If the improved photometric methods practised at the Harvard observatory make it possible to observe these with sufficient accuracy, the difference in the results found for the velocity of light when Jupiter is nearest to and farthest from the line of motion will give, not merely the motion of the solar system with reference to the stars, but with reference to the luminiferous æther itself.

4. ON THE LAWS OF THE REFLEXION AND REFRACTION OF LIGHT AT THE COMMON SURFACE OF TWO NON-CRYSTALLIZED MEDIA*

G. GREEN

M. CAUCHY seems to have been the first who saw fully the utility of applying to the Theory of Light those formulæ which represent the motions of a system of molecules acting on each other by mutually attractive and repulsive forces; supposing always that in the mutual action of any two particles, the particles may be regarded as points animated by forces directed along the right line which joins them. This last supposition, if applied to those compound particles, at least, which are separable by mechanical division, seems rather restrictive; as many phenomena, those of crystallization for instance, seem to indicate certain polarities in these particles. If, however, this were not the case, we are so perfectly ignorant of the mode of action of the elements of the luminiferous ether on each other, that it would seem a safer method to take some general physical principle as the basis of our reasoning, rather than assume certain modes of action, which, after all, may be widely different from the mechanism employed by nature; more especially if this principle include in itself, as a particular case, those before used by M. Cauchy and others, and also lead to a much more simple process of calculation. The principle selected

* *Trans. Camb. Phil. Soc.* **7**, 1, 113 (1838).

as the basis of the reasoning contained in the following paper is this: In whatever way the elements of any material system may act upon each other, if all the internal forces exerted be multiplied by the elements of their respective directions, the total sum for any assigned portion of the mass will always be the exact differential of some function. But, this function being known, we can immediately apply the general method given in the *Mécanique Analytique*, and which appears to be more especially applicable to problems that relate to the motions of systems composed of an immense number of particles mutually acting upon each other. One of the advantages of this method, of great importance, is, that we are necessarily led by the mere process of the calculation, and with little care on our part, to all the equations and conditions which are *requisite* and *sufficient* for the complete solution of any problem to which it may be applied.

The present communication is confined almost entirely to the consideration of non-crystallized media; for which it is proved, that the function due to the molecular actions, in its most general form, contains only two arbitrary coefficients, A and B; the values of which depend of course on the unknown internal constitution of the medium under consideration, and it would be easy to shew, for the most general case, that any arbitrary disturbance, excited in a very small portion of the medium, would in general give rise to two spherical waves, one propagated entirely by normal, the other entirely by transverse, vibrations, and such that if the velocity of transmission of the former wave be represented by $\sqrt{A}$, that of the latter would be represented by $\sqrt{B}$. But in the transmission of light through a prism, though the wave which is propagated by normal vibrations were incapable itself of affecting the eye, yet it would be capable of giving rise to an ordinary wave of light propagated by transverse vibrations, except in the extreme cases where $A/B = 0$, or $A/B =$ a very large quantity; which, for the sake of simplicity, may be regarded as infinite; and it is not difficult to prove that the equilibrium of our medium would be unstable

unless $A/B > \frac{4}{3}$. We are therefore compelled to adopt the latter value of A/B, and thus to admit that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations is very great compared with that of ordinary light.

The principal results obtained in this paper relate to the intensity of the wave reflected at the common surface of two media, both for light polarized in and perpendicular to the plane of incidence; and likewise to the change of phase which takes place when the reflexion becomes total. In the former case, our values agree precisely with those given by Fresnel; supposing, as he has done, that the direction of the actual motion of the particles of the luminiferous ether is perpendicular to the plane of polarization. But it results from our formulæ, when the light is polarized perpendicular to the plane of incidence, that the expressions given by Fresnel are only very near approximations; and that the intensity of the reflected wave will never become absolutely null, but only attain a minimum value; which, in the case of reflexion from water at the proper angle, is $\frac{1}{151}$ part of that of the incident wave. This minimum value increases rapidly, as the index of refraction increases, and thus the quantity of light reflected at the polarizing angle, becomes considerable for highly refracting substances, a fact which has been long known to experimental philosophers.

It may be proper to observe, that M. Cauchy (*Bulletin des Sciences*, 1830) has given a method of determining the intensity of the waves reflected at the common surface of two media. He has since stated (*Nouveaux Exercises des Mathématiques*) that the hypothesis employed on that occasion is inadmissible, and has promised in a future memoir, to give a *new mechanical principle* applicable to this and other questions; but I have not been able to learn whether such a memoir has yet appeared. The first method consisted in satisfying a part, and only a part, of the conditions belonging to the surface of junction, and the consideration of the waves propagated by normal vibrations was wholly overlooked, though it is easy to perceive, that in general waves of this kind must necessarily be produced when the incident wave is polarized perpendicular to the plane of incidence, in consequence of the incident and refracted waves being in different planes. Indeed, without introducing the consideration of these last waves, it is impossible to satisfy the whole of the conditions due to the surface of junction of the two media. But when this consideration is introduced, the whole of the conditions may be satisfied, and the principles given in the *Mécanique Analytique* became abundantly sufficient for the solution of the problem.

In conclusion, it may be observed, that the radius of the sphere of sensible action of the molecular forces has been regarded as insensible with respect to the length λ of a wave of light, and thus, for the sake of simplicity, certain terms have been disregarded on which the different refrangibility of differently coloured rays might be supposed to depend. These terms, which are necessary to be considered when we are treating of the dispersion, serve only to render our formulæ uselessly complex in other investigations respecting the phenomena of light.

Let us conceive a mass composed of an immense number of molecules acting on each other by any kind of molecular forces, but which are sensible only at insensible distances, and let moreover the whole system be quite free from all extraneous action of every kind. Then x, y and z being the co-ordinates of any particle of the medium under consideration when in equilibrium, and

$$x+u, \quad y+v, \quad z+w,$$

the co-ordinates of the same particle in a state of motion (where u, v, and w are very small functions of the original co-ordinates (x, y, z), of any particle and of the time (t)), we get, by combining D'Alembert's principle with that of virtual velocities,

$$\Sigma Dm\left\{\frac{d^2u}{dt^2}\,\delta u+\frac{d^2v}{dt^2}\,\delta v+\frac{d^2w}{dt^2}\,\delta w\right\} = \Sigma Dv\cdot\delta\phi \tag{1}$$

Dm, and Dv being exceedingly small corresponding elements of the mass and volume of the medium, but which nevertheless contain a very great number of molecules, and $\delta\phi$ the exact differential of some function and entirely due to the internal actions of the particles of the medium on each other. Indeed, if $\delta\phi$ were not an exact differential, a perpetual motion would be possible, and we have every reason to think, that the forces in nature are so disposed as to render this a natural impossibility.

Let us now take any element of the medium, rectangular in a state of repose, and of which the sides are dx, dy, dz; the length of the sides composed of the same particles will in a state of motion become

$$dx' = dx(1+s_1), \quad dy' = dy(1+s_2), \quad dz' = dz(1+s_3);$$

where s_1, s_2, s_3 are exceedingly small quantities of the first order. If, moreover, we make,

$$\alpha = \cos\angle\frac{dy'}{dz'}, \quad \beta = \cos\angle\frac{dx'}{dz'}, \quad \gamma = \cos\angle\frac{dx'}{dy'};$$

α, β, and γ will be very small quantities of the same order. But, whatever may be the nature of the internal actions, if we represent by

$$\delta\phi\, dx\, dy\, dz,$$

the part of the second member of the equation (1), due to the molecules in the element under consideration, it is evident, that ϕ will remain the same when all the sides and all the angles of the parallelpiped, whose sides are dx' dy' dz', remain unaltered, and therefore its most general value must be of the form

$$\phi = \text{function}\,\{s_1, s_2, s_3, \alpha, \beta, \gamma\}.$$

But $s_1, s_2, s_3, \alpha, \beta, \gamma$ being very small quantities of the first order, we may expand ϕ in a very convergent series of the form

$$\phi = \phi_0+\phi_1+\phi_2+\phi_3+\&c.:$$

ϕ_0, ϕ_1, ϕ_2, &c. being homogeneous functions of the six quantities α, β, γ, s_1, s_2, s_3 of the degrees 0, 1, 2, &c. each of which is very great compared with the next following one. If now, ϱ represent the primitive density of the element $dx\,dy\,dz$, we may write $\varrho\,dx\,dy\,dz$ in the place of Dm in the formula (1), which will thus become, since ϕ_2 is constant,

$$\iiint \varrho\,dx\,dy\,dz\left\{\frac{d^2u}{dt^2}\delta u+\frac{d^2v}{dt^2}\delta v+\frac{d^2w}{dt^2}\delta w\right\}$$
$$=\iiint dx\,dy\,dz(\delta\phi_1+\delta\phi_2+\text{\&c.});$$

the triple integrals extending over the whole volume of the medium under consideration.

But by the supposition, when $u = 0$, $v = 0$ and $w = 0$, the system is in equilibrium, and hence

$$0=\iiint dx\,dy\,dz\,\delta\phi_1:$$

seeing that ϕ_1 is a homogeneous function of s_1, s_2, s_3, α, β, γ of the *first* degree only. If therefore we neglect ϕ_3, ϕ_4, &c. which are exceedingly small compared with ϕ_2, our equation becomes

$$\iiint \varrho\,dx\,dy\,dz\left\{\frac{d^2u}{dt^2}\delta u+\frac{d^2v}{dt^2}\delta v+\frac{d^2w}{dt^2}\delta w\right\}$$
$$=\iiint dx\,dy\,dz\,\delta\phi_2; \qquad (2)$$

the integrals extending over the whole volume under consideration. The formula just found is true for any number of media comprised in this volume, provided the whole system be perfectly free from all extraneous forces, and subject only to its own molecular actions.

If now we can obtain the value of ϕ_2, we shall only have to apply the general methods given in the *Mécanique Analytique*.

But ϕ_2, being a homogeneous function of six quantities of the second degree, will in its most general form contain 21 arbitrary coefficients. The proper value to be assigned to each will of course depend on the internal constitution of the medium. If, however, the medium be a non-crystallized one, the form of ϕ_2 will remain the same, whatever be the directions of the co-ordinate axes in space. Applying this last consideration, we shall find that the most general form of ϕ_2 for non-crystallized bodies contains only two arbitrary coefficients. In fact, by neglecting quantities of the higher orders, it is easy to perceive that

$$s_1=\frac{du}{dx},\quad s_2=\frac{dv}{dy},\quad s_3=\frac{dw}{dz},$$

$$\alpha=\frac{dw}{dy}+\frac{dv}{dz},\quad \beta=\frac{dw}{dx}+\frac{du}{dz},\quad \gamma=\frac{du}{dy}+\frac{dv}{dx},$$

and if the medium is symmetrical with regard to the plane (xy) only, ϕ_2 will remain unchanged when $-z$ and $-w$ are written for z and w. But this alteration evidently changes α and β to $-\alpha$ and $-\beta$. Similar observations apply to the planes (xz) (yz). If therefore the medium is merely symmetrical with respect to each of the three co-ordinate planes, we see that ϕ_2 must remain unaltered when

$$\left.\begin{array}{ll}\text{or} & -z,\ -w,\ -\alpha,\ -\beta\\ \text{or} & -y,\ -v,\ -\alpha,\ -\gamma\\ \text{or} & -x,\ -u,\ -\beta,\ -\gamma\end{array}\right\}\ \text{are written for}\ \left\{\begin{array}{l} z,\ w,\ \alpha,\ \beta\\ y,\ v,\ \alpha,\ \gamma\\ x,\ u,\ \beta,\ \gamma.\end{array}\right.$$

In this way the 21 coefficients are reduced to 9, and the resulting function is of the form

$$G\left(\frac{du}{dx}\right)^2+H\left(\frac{dv}{dy}\right)^2+I\left(\frac{dw}{dz}\right)^2+L\alpha^2+M\beta^2+N\gamma^2$$
$$+2P\frac{dv}{dy}\cdot\frac{dw}{dz}+2Q\frac{du}{dx}\cdot\frac{dw}{dz}+2R\frac{du}{dx}\cdot\frac{dv}{dy}=\phi_2\ldots(A).$$

Probably the function just obtained may belong to those crystals which have three axes of elasticity at right angles to each other.

Suppose now we further restrict the generality of our function by making it symmetrical all round one axis, as that of z for instance. By shifting the axis of x through the infinitely small angle $\delta\theta$,

$$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} \quad \text{becomes} \quad \left\{\begin{matrix} x+y\,\delta\theta \\ y-x\,\delta\theta, \\ z \end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{d}{dx} \\[2ex] \dfrac{d}{dy} \\[2ex] \dfrac{d}{dz} \end{matrix}\right\} \quad \text{becomes} \quad \left\{\begin{matrix} \dfrac{d}{dx}+\delta\theta\dfrac{d}{dy} \\[2ex] \dfrac{d}{dy}-d\theta\dfrac{d}{dx}, \\[2ex] \dfrac{d}{dz} \end{matrix}\right.$$

and

$$\left.\begin{matrix} u \\ v \\ w \end{matrix}\right\} \quad \text{becomes} \quad \left\{\begin{matrix} u+v\,\delta\theta \\ v-u\,\delta\theta. \\ w \end{matrix}\right.$$

Making these substitutions in (A), we see that the form of ϕ_2 will not remain the same for the new axes, unless

$$G = H = 2N+R,$$
$$L = M,$$
$$P = Q;$$

and thus we get

$$\phi_2 = G\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2\right\}+I\left(\frac{dw}{dz}\right)^2+L(\alpha^2+\beta^2)$$
$$+N\gamma^2+2P\left(\frac{dv}{dy}+\frac{du}{dx}\right)\frac{dw}{dz}+(2G-4N)\frac{du}{dx}\cdot\frac{dv}{dy}\ \ldots\ (B);$$

under which form it may possibly be applied to uniaxal crystals.

Lastly, if we suppose the function ϕ_2 symmetrical with respect to all three axes, there results

$$G = H = I = 2N+R,$$
$$L = M = N,$$
$$P = Q = R;$$

and consequently,

$$\phi_2 = G\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dy}\right)^2+\left(\frac{dw}{dz}\right)^2\right\}+L(\alpha^2+\beta^2+\gamma^2)$$
$$+(2G-4L)\left\{\frac{dv}{dy}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dv}{dy}\right\};$$

or, by merely changing the two constants and restoring the values of α, β, and γ,

$$2\phi_2 = -A\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2$$
$$-B\left\{\left(\frac{du}{dy}+\frac{dv}{dx}\right)^2+\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2+\left(\frac{dv}{dz}+\frac{dw}{dy}\right)^2\right.$$
$$\left.-4\left(\frac{dv}{dy}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dw}{dz}+\frac{du}{dx}\cdot\frac{dv}{dy}\right)\right\}\ \ldots\ (C).$$

This is the most general form that ϕ_2 can take for non-crystallized bodies, in which it is perfectly indifferent in what directions the rectangular axes are placed. The same result might be obtained from the most general value of ϕ_2, by the method before used to make ϕ_2 symmetrical all round the axis of z, applied also to the other two axes. It was, indeed, thus I first obtained it. The method given in the text, however, and which is very similar to one used by M. Cauchy, is not only more simple, but has the advantage of furnishing two intermediate results, which may possibly be of use on some future occasion.

Let us now consider the particular case of two indefinitely extended media, the surface of junction when in equilibrium being a plane of infinite extent, horizontal (suppose), and which we shall take as that of (yz), and conceive the axis of x positive directed downwards. Then if ϱ be the constant density of the upper, and $\varrho_{,}$ that of the lower medium, ϕ_2 and $\phi_2^{(1)}$ the corresponding functions due to the molecular actions; the equation (2) adapted to the present case will become

$$\iiint \varrho\, dx\, dy\, dz \left\{\frac{d^2u}{dt^2}\,\delta u+\frac{d^2v}{dt^2}\,\delta v+\frac{d^2w}{dt^2}\,\delta w\right\}$$

$$+\iiint \varrho_{,}\, dx\, dy\, dz \left\{\frac{d^2u_{,}}{dt^2}\,\delta u_{,}+\frac{d^2v_{,}}{dt^2}\,\delta v_{,}+\frac{d^2w_{,}}{dt^2}\,dw_{,}\right\},$$

$$=\iiint dx\, dy\, dz\, \phi_2+\iiint dx\, dy\, dz\, \phi_2^{(1)}; \qquad (3)$$

$u_{,}$, $v_{,}$, $w_{,}$ belonging to the lower fluid, and the triple integrals being extended over the whole volume of the fluids to which they respectively belong.

It now only remains to substitute for ϕ_2 and $\phi_2^{(1)}$ their values, to effect the integrations by parts, and to equate separately to zero the coefficients of the independent variations. Substituting therefore for ϕ_2 its value (C), we get

$$\iiint dx\, dy\, dz\, \delta\phi_2$$

$$=-A\iiint dx\, dy\, dz\left\{\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\left(\frac{d\delta u}{dx}+\frac{d\delta v}{dy}+\frac{d\delta w}{dz}\right)\right\}$$

$$-B\iiint dx\, dy\, dz\left\{\left(\frac{du}{dy}+\frac{dv}{dx}\right)\left(\frac{d\delta u}{dy}+\frac{d\delta v}{dx}\right)\right.$$

$$+\left(\frac{du}{dz}+\frac{dw}{dx}\right)\left(\frac{d\delta u}{dz}+\frac{d\delta w}{dx}\right)+\left(\frac{dv}{dz}+\frac{dw}{dy}\right)\left(\frac{d\delta v}{dz}+\frac{d\delta w}{dy}\right)$$

$$-2\left[\left(\frac{dv}{dy}\cdot\frac{d\delta w}{dz}+\frac{dw}{dz}\cdot\frac{d\delta v}{dy}\right)+\left(\frac{du}{dx}\cdot\frac{d\delta w}{dz}+\frac{dw}{dz}\cdot\frac{d\delta u}{dx}\right)\right.$$

$$\left.\left.+\left(\frac{du}{dx}\cdot\frac{d\delta v}{dy}+\frac{dv}{dy}\cdot\frac{d\delta u}{dx}\right)\right]\right\}$$

$$=-\iint dy\, dz\left\{A\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)-2B\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}\cdot\delta u$$

$$-\iint dy\, dz\left\{B\left(\frac{du}{dy}+\frac{dv}{dx}\right)\delta v+B\left(\frac{du}{dz}+\frac{dw}{dx}\right)\delta w\right\}$$

$$+\iiint dx\, dy\, dz\left\{A\frac{d}{dx}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right.$$

$$\left.+B\left[\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}-\frac{d}{dx}\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right]\right\}\cdot\delta u$$

$$+\left\{A\frac{d}{dy}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+B\left[\frac{d^2v}{dx^2}+\frac{d^2v}{dz^2}\right.\right.$$

$$\left.\left.-\frac{d}{dy}\left(\frac{du}{dx}+\frac{dw}{dz}\right)\right]\right\}\delta v+\left\{A\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\right.$$

$$\left.+B\left[\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}-\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}\right)\right]\right\}\delta w;$$

seeing that we may neglect the double integrals at the limits $x=-\infty$, $y=\pm\infty$, $z=\pm\infty$; as the conditions imposed at these limits cannot affect the motion of the system at any *finite* distance from the origin; and thus the double integrals belong only to the surface of junction, of which the equation, in a state of equilibrium, is

$$0=x.$$

In like manner we get

$$\iiint dx\, dy\, dz\, \delta\phi_2^{(1)}$$

$$=+\iint dy\, dz\left\{A_{,}\left(\frac{du_{,}}{dx}+\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)-2B_{,}\left(\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)\right\}\delta u_{,}$$

$$+\iint dy\, dz\left\{B\left(\frac{du_{,}}{dy}+\frac{dv_{,}}{dx}\right)\delta v_{,}+B\left(\frac{du_{,}}{dz}+\frac{dw_{,}}{dx}\right)\delta w_{,}\right\}$$

+ the triple integral;

since it is the *least* value of x which belongs to the surface of junction in the *lower* medium, and therefore the double integrals belonging to the limiting surface must have their signs changed.

If, now, we substitute the preceding expression in (3), equate separately to zero the coefficients of the independent variation δu, δv, δw, under the triple sign of integration, there results for the upper medium

$$\begin{aligned}
\varrho\frac{d^2u}{dt^2} &= A\frac{d}{dx}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\\
&\quad +B\left\{\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}-\frac{d}{dx}\cdot\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right\};\\
\varrho\frac{d^2v}{dt^2} &= A\frac{d}{dy}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\\
&\quad +B\left\{\frac{d^2v}{dx^2}+\frac{d^2v}{dz^2}-\frac{d}{dy}\cdot\left(\frac{du}{dx}+\frac{dw}{dz}\right)\right\}; \qquad (4)\\
\varrho\frac{d^2w}{dt^2} &= A\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)\\
&\quad +B\left\{\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}-\frac{d}{dz}\cdot\left(\frac{du}{dx}+\frac{dv}{dy}\right)\right\};
\end{aligned}$$

and by equating the coefficients of $\delta u_{,}$, $\delta v_{,}$, $\delta w_{,}$, we get three similar equations for the lower medium.

To the six general equations just obtained, we must add the conditions due to the surface of junction of the two media; and at this surface we have first,

$$u = u_{,}, \quad v = v_{,}, \quad w = w_{,}, \qquad (\text{when } x = 0); \qquad (5)$$

and consequently,

$$\delta u = \delta u_{,}; \qquad \delta v = \delta v_{,}; \qquad \delta w = \delta w_{,}.$$

But the part of the equation (3) belonging to this surface, and which yet remains to be satisfied, is

$$\begin{aligned}
0 = &-\iint dy\,dz\left\{A\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)-2B\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\right\}\delta u\\
&+\iint dy\,dz\left\{A\left(\frac{du_{,}}{dx}+\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)-2B_{,}\left(\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)\right\}\delta u_{,}\\
&+\iint dy\,dz\left\{B\left(\frac{du}{dy}+\frac{dv}{dx}\right)\delta v+B\left(\frac{du}{dz}+\frac{dw}{dx}\right)\delta w\right\}\\
&+\iint dy\,dz\left\{B_{,}\left(\frac{du_{,}}{dy}+\frac{dv_{,}}{dx}\right)\delta v_{,}+B_{,}\left(\frac{du_{,}}{dz}+\frac{dw_{,}}{dx}\right)\delta w_{,}\right\};
\end{aligned}$$

and as $\delta u = \delta u_{,}$, &c., we obtain, as before,

$$\begin{aligned}
&A\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)-2B\left(\frac{dv}{dy}+\frac{dw}{dz}\right)\\
&\quad = A_{,}\left(\frac{du_{,}}{dx}+\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)-2B_{,}\left(\frac{dv_{,}}{dy}+\frac{dw_{,}}{dz}\right)\\
&B\left(\frac{du}{dy}+\frac{dv}{dx}\right) = B_{,}\left(\frac{du_{,}}{dy}+\frac{dv_{,}}{dx}\right), \qquad (6)\\
&B\left(\frac{du}{dz}+\frac{dw}{dx}\right) = B_{,}\left(\frac{du_{,}}{dz}+\frac{dw_{,}}{dx}\right);
\end{aligned}$$

and these belong to the particular value $x = 0$.

The six particular conditions (5) and (6), belonging to the surface of junction of the two media, combined with the six general equations before obtained, are *necessary* and *sufficient* for the complete determination of the motion of the two media, supposing the initial state of each given. We shall not here attempt their general solution, but merely consider the propagation of a plane wave of infinite extent, accompanied by its reflected and refracted waves, as in the preceding paper on Sound.

Let the direction of the axis of z, which yet remains arbitrary, be taken parallel to the intersection of the plane of the incident wave with the surface of junction, and suppose the disturbance of the particles to be wholly in the direction of the axis of z, which is the case with light polarized in the plane of incidence, according to Fresnel. Then we have

$$0 = u, \quad 0 = v, \quad 0 = u_{,}, \quad 0 = v_{,};$$

and supposing the disturbance the same for every point of the same front of a wave, w and $w_{,}$ will be independent of z, and thus the three general equations (4) will all be satisfied if

$$\varrho \frac{d^2w}{dt^2} = B\left\{\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right\},$$

or by making $B/\varrho = \gamma^2$,

$$\frac{d^2w}{dt^2} = \gamma^2\left\{\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right\}. \tag{7}$$

Similarly in the lower medium we have

$$\frac{d^2w_{,}}{dt^2} = \gamma_{,}^2\left\{\frac{d^2w_{,}}{dx^2} + \frac{d^2w_{,}}{dy^2}\right\}, \tag{8}$$

$w_{,}$ and $\gamma_{,}$ belonging to this medium.

It now remains to satisfy the conditions (5) and (6). But these are all satisfied by the preceding values provided

$$w = w_{,},$$

$$B\frac{dw}{dx} = B_{,}\frac{dw_{,}}{dx}.$$

The formulæ which we have obtained are quite general, and will apply to the ordinary elastic fluids by making $B=0$. But for all the known gases, A is independent of the nature of the gas, and consequently $A=A_{,}$. If, therefore, we suppose $B=B_{,}$, at least when we consider those phenomena only which depend merely on

different states of the same medium, as is the case with light, our conditions become†

$$\left.\begin{aligned} w &= w_{,}; \\ \frac{dw}{dx} &= \frac{dw_{,}}{dx} \end{aligned}\right\} \quad \text{(when } x = 0\text{)}. \tag{9}$$

The disturbance in the upper medium which contains the incident and reflected wave, will be represented, as in the case of Sound, by

$$w = f(ax+by+ct)+F(-ax+by+ct);$$

f belonging to the incident, F to the reflected plane wave, and c being a negative quantity. Also in the lower medium,

$$w_{,} = f_{,}(a_{,}x+by+ct).$$

These values evidently satisfy the general equations (7) and (8), provided $c^2 = \gamma^2(a^2+b^2)$, and $c^2 = \gamma_{,}^2(a_{,}^2+b^2)$; we have therefore only to satisfy the conditions (9), which give

$$f(by+ct)+F(by+ct) = f_{,}(by+ct),$$

$$af'(by+ct)-aF'(by+ct) = a_{,}f_{,}'(by+ct).$$

Taking now the differential coefficient of the first equation, and writing to abridge the characteristics of the functions only, we get

$$2f' = \left(1+\frac{a_{,}}{a}\right)f_{,}', \quad \text{and} \quad 2F' = \left(1-\frac{a_{,}}{a}\right)f_{,}',$$

† Though for all known gases A is independent of the nature of the gas, perhaps it is extending the analogy rather too far, to assume that in the luminiferous ether the constants A and B must always be independent of the state of the ether, as found in different refracting substances. However, since this hypothesis greatly simplifies the equations due to the surface of junction of the two media, and is itself the most simple that could be selected, it seemed natural first to deduce the consequences which follow from it before trying a more complicated one, and, as far as I have yet found, these consequences are in accordance with observed facts.

and therefore

$$\frac{F'}{f'} = \frac{1-\dfrac{a_{\prime}}{a}}{1+\dfrac{a_{\prime}}{a}} = \frac{a-a_{\prime}}{a+a_{\prime}} = \frac{\cot\theta-\cot\theta_{\prime}}{\cot\theta+\cot\theta_{\prime}} = \frac{\sin(\theta_{\prime}-\theta)}{\sin(\theta_{\prime}+\theta)};$$

θ and $\theta_{\prime}$ being the angles of incidence and refraction.

This ratio between the intensity of the incident and reflected waves is exactly the same as that for light polarized in the plane of incidence (vide Airy's *Tracts*, p. 356†), and which Fresnel supposes to be propagated by vibrations perpendicular to the plane of incidence, agreeably to what has been assumed in the foregoing process.

We will now limit the generality of the functions f, F and $f_{\prime}$, by supposing the law of the motion to be similar to that of a cycloidal pendulum; and if we farther suppose the angle of incidence to be increased until the refracted wave ceases to be transmitted in the regular way, as in our former paper on Sound, the proper integral of the equation

$$\frac{d^2w_{\prime}}{dt^2} = \gamma_{\prime}^2\left\{\frac{d^2w_{\prime}}{dx^2}+\frac{d^2w_{\prime}}{dy^2}\right\}$$

will be

$$w_{\prime} = \varepsilon^{-a_{\prime}'x}B\sin\psi; \tag{10}$$

where $\psi = by+ct$, and $a_{\prime}'$ is determined by

$$\gamma_{\prime}^2(b^2-a_{\prime}'^2) = c^2 = \gamma^2(b^2+a^2). \tag{11}$$

But one of the conditions (9) will introduce *sines* and the other *cosines*, in such a way that it will be impossible to satisfy them unless we introduce both *sines* and *cosines* into the value of w, or, which amounts to the same, unless we make

$$w = \alpha\sin(ax+by+ct+e)+\beta\sin(-ax+by+ct+e_{\prime}) \tag{12}$$

in the first medium, instead of

$$w = \alpha\sin(ax+by+ct)+\beta\sin(-ax+by+ct),$$

† [Airy on the Undulatory Theory of Optics, p. 109, Art. 128.]

which would have been done had the refracted wave been transmitted in the usual way, and consequently no exponential been introduced into the value of $w_{\prime}$. We thus see the analytical reason for what is called the change of phase which takes place when the reflexion of light becomes total.

Substituting now (10) and (12), in the equations (9), and proceeding precisely as for Sound, we get

$$0 = \alpha\cos e-\beta\cos e_{\prime},$$
$$0 = \alpha\sin e+\beta\sin e_{\prime},$$
$$\frac{a_{\prime}'}{a}B = \alpha\sin e-\beta\sin e_{\prime},$$
$$B = \alpha\cos e+\beta\cos e_{\prime}.$$

Hence there results $\alpha = \beta$, and $e_{\prime} = -e$, and

$$\tan e = \frac{a_{\prime}'}{a} = \frac{a_{\prime}'}{b} \div \frac{a}{b} = \frac{a_{\prime}'}{b}\tan\theta.$$

But by (11),

$$\frac{a_{\prime}'}{b} = \sqrt{\left\{1-\frac{\gamma^2}{\gamma_{\prime}^2}\left(1+\frac{a^2}{b^2}\right)\right\}} = \sqrt{\left(1-\frac{1}{\mu^2\sin^2\theta}\right)};$$

by introducing μ the index of refraction, and θ the angle of incidence. Thus,

$$\tan e = \frac{\sqrt{(\mu^2\sin^2\theta-1)}}{\mu\cos\theta};$$

and as e represents half the alteration of phase in passing from the incident to the reflected wave, we see that here also our result agrees precisely with Fresnel's for light polarized in the plane of incidence. (Vide Airy's *Tracts*, p. 362.†)

Let us now conceive the direction of the transverse vibrations in the incident wave to be perpendicular to the direction in the case

† [Airy, *ubi sup.* p. 114, Art. 133.]

just considered; and therefore that the actual motions of the particles are all parallel to the intersection of the plane of incidence (xy) with the front of the wave. Then, as the planes of the incident and refracted waves do not coincide, it is easy to perceive that at the surface of junction there will, in this case, be a resolved part of the disturbance in the direction of the normal; and therefore, besides the incident wave, there will, in general, be an accompanying reflected and refracted wave, in which the vibrations are transverse, and another pair of accompanying reflected and refracted waves, in which the directions of the vibrations are normal to the fronts of the waves. In fact, unless the consideration of the two latter waves is also introduced, it is impossible to satisfy all the conditions at the surface of junction; and these are as essential to the complete solution of the problem, as the general equations of motion.

The direction of the disturbance being in plane (xy) $w=0$, and as the disturbance of every particle in the same front of a wave is the same, u and v are independent of z. Hence, the general equations (4) for the first medium become

$$\frac{d^2u}{dt^2} = g^2\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\gamma^2\frac{d}{dy}\left(\frac{du}{dy}-\frac{dv}{dx}\right),$$

$$\frac{d^2v}{dt^2} = g^2\frac{d}{dy}\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\gamma^2\frac{d}{dx}\left(\frac{dv}{dx}-\frac{du}{dy}\right),$$

where $g^2 = \dfrac{A}{\varrho}$, and $\gamma^2 = \dfrac{B}{\varrho}$.

These equations might be immediately employed in their present form; but they will take a rather more simple form, by making

$$\left.\begin{aligned} u &= \frac{d\phi}{dx}+\frac{d\psi}{dy} \\ v &= \frac{d\phi}{dy}-\frac{d\psi}{dx}\end{aligned}\right\}; \tag{13}$$

ϕ and ψ being two functions of x, y, and t, to be determined.

By substitution, we readily see that the two preceding equations are equivalent to the system

$$\left.\begin{aligned} \frac{d^2\phi}{dt^2} &= g^2\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right) \\ \frac{d^2\psi}{dt^2} &= \gamma^2\left(\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2}\right)\end{aligned}\right\}. \tag{14}$$

In like manner, if in the second medium we make

$$\left.\begin{aligned} u_{,} &= \frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy} \\ v_{,} &= \frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx}\end{aligned}\right\}, \tag{15}$$

we get to determine $\phi_{,}$ and $\psi_{,}$ the equations

$$\left.\begin{aligned} \frac{d^2\phi_{,}}{dt^2} &= g_{,}^2\left(\frac{d^2\phi_{,}}{dx^2}+\frac{d^2\phi_{,}}{dy^2}\right) \\ \frac{d^2\psi_{,}}{dt^2} &= \gamma_{,}^2\left(\frac{d^2\psi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dy^2}\right)\end{aligned}\right\} \tag{16}$$

and as we suppose the constants A and B the same for both media, we have

$$\frac{\gamma}{\gamma_{,}} = \frac{g}{g_{,}}.$$

For the complete determination of the motion in question, it will be necessary to satisfy all the conditions due to the surface of junction of the two media. But, since $w = 0$ and $w_{,} = 0$, also, since u, v, $u_{,}$, $v_{,}$ are independent of z, the equations (5) and (6) become

$$u = u_{,}, \quad v = v_{,};$$

$$A\left(\frac{du}{dx}+\frac{dv}{dy}\right)-2B\frac{dv}{dy} = A\left(\frac{du_{,}}{dx}+\frac{dv_{,}}{dy}\right)-2B\frac{dv_{,}}{dy},$$

$$\frac{du}{dy}+\frac{dv}{dx} = \frac{du_{,}}{dy}+\frac{dv_{,}}{dx},$$

provided $x = 0$. But since $x = 0$ in the last equations, we many differentiate them with regard to any of the independent variables except x, and thus the two latter, in consequence of the two former, will become

$$\frac{du}{dx} = \frac{du_{,}}{dx}, \quad \frac{dv}{dx} = \frac{dv_{,}}{dx}.$$

Substituting now for u, v, &c., their values (13) and (15), in ϕ and ψ, the four resulting conditions relative to the surface of junction of the two media may be written,

$$\left.\begin{aligned} \frac{d\phi}{dx}+\frac{d\psi}{dy} &= \frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy} \\ \frac{d\phi}{dy}-\frac{d\psi}{dx} &= \frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx} \\ \frac{d^2\phi}{dx^2}+\frac{d^2\psi}{dx\,dy} &= \frac{d^2\phi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dx\,dy} \\ \frac{d^2\phi}{dx\,dy}-\frac{d^2\psi}{dx^2} &= \frac{d^2\phi_{,}}{dx\,dy}-\frac{d^2\psi_{,}}{dx^2} \end{aligned}\right\} \text{(when } x = 0\text{);}$$

or since we may differentiate with respect to y, the first and fourth equations give

$$\frac{d^2\psi}{dx^2}+\frac{d^2\psi}{dy^2} = \frac{d^2\psi_{,}}{dx^2}+\frac{d^2\psi_{,}}{dy^2};$$

in like manner, the second and third give

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2} = \frac{d^2\phi_{,}}{dx^2}+\frac{d^2\phi_{,}}{dy^2},$$

which, in consequence of the general equations (14) and (16), become

$$\frac{d^2\psi}{\gamma^2\,dt^2} = \frac{d^2\psi_{,}}{\gamma_{,}^2\,dt^2}, \quad \text{and} \quad \frac{d^2\phi}{g^2\,dt^2} = \frac{d^2\phi_{,}}{g_{,}^2\,dt^2}.$$

Hence, the equivalent of the four conditions relative to the surface of junction may be written

$$\left.\begin{aligned} \frac{d\phi}{dx}+\frac{d\psi}{dy} &= \frac{d\phi_{,}}{dx}+\frac{d\psi_{,}}{dy} \\ \frac{d\phi}{dy}-\frac{d\psi}{dx} &= \frac{d\phi_{,}}{dy}-\frac{d\psi_{,}}{dx} \\ \frac{d^2\phi}{g^2\,dt^2} &= \frac{d^2\phi_{,}}{g_{,}^2\,dt^2} \\ \frac{d^2\psi}{\gamma^2\,dt^2} &= \frac{d^2\psi_{,}}{\gamma_{,}^2\,dt^2} \end{aligned}\right\} \text{(when } x = 0\text{).} \qquad (17)$$

If we examine the expressions (13) and (15), we shall see that the disturbances due to ϕ and $\phi_{,}$ are normal to the front of the wave to which they belong, whilst those which are due to ψ, $\psi_{,}$ are transverse or wholly in the front of the wave. If the coefficients A and B did not differ greatly in magnitude, waves propagated by both kinds of vibrations must in general exist, as was before observed. In this case, we should have in the upper medium

$$\left.\begin{aligned} \psi &= f(ax+by+ct)+F(-ax+by+ct) \\ \text{and} \quad \phi &= \chi_{,}(-a'x+by+ct) \end{aligned}\right\}; \qquad (18)$$

and for the lower one

$$\left.\begin{aligned} \psi_{,} &= f_{,}(a_{,}x+by+ct) \\ \phi_{,} &= \chi_{,}(a_{,}'x+by+ct) \end{aligned}\right\}. \qquad (19)$$

The coefficients b and c being the same for all the functions to simplify the results, since the indeterminate coefficients $a'_{,}$ $a_{,}$ a' will allow the fronts of the waves to which they respectively belong, to take any position that the nature of the problem may require. The coefficient of x in F belonging to that reflected wave, which, like the incident one, is propagated by transverse vibrations would have been determined exactly like $a'_{,}$ $a_{,}$ a', as, however, it evidently $= -a$, it was for the sake of simplicity introduced immediately into our formulæ.

By substituting the values just given in the general equations (14) and (16), there results

$$c^2 = (a^2+b^2)\gamma^2 = (a_{,}^2+b^2)\gamma_{,}^2 = (a'^2+b^2)g^2 = (a_{,}'^2+b^2)g'^2,$$

we have thus the position of the fronts of the reflected and refracted waves.

It now remains to satisfy the conditions due to the surface of junction of the two media. Substituting, therefore, the values (18) and (19) in the equations (17), we get

$$f''+F'' = \frac{\gamma^2}{\gamma_{,}^2} f_{,}'',$$

$$\chi'' = \frac{g^2}{g_{,}^2} \chi_{,}'';$$

$$-a'\chi'+b(f'+F') = a_{,}'\chi_{,}'+bf_{,}',$$
$$b\chi'-a(f'-F') = b\chi_{,}'-a_{,}f_{,}';$$

where to abridge, the characteristics only of the functions are written.

By means of the last four equations, we shall readily get the values of F'' χ'' $f_{,}''$ $\chi_{,}''$ in terms of f'', and thus obtain the intensities of the two reflected and two refracted waves, when the coefficients A and B do not differ greatly in magnitude, and the angle which the incident wave makes with the plane surface of junction is contained within certain limits. But in the introductory remarks, it was shewn that $A/B =$ a very great quantity which may be regarded as infinite, and therefore g and $g_{,}$ may be regarded as infinite compared with γ and $\gamma_{,}$. Hence, for all angles of incidence except such as are infinitely small, the waves dependent on ϕ and $\phi_{,}$ cease to be transmitted in the regular way. We shall therefore, as before, restrain the generality of our functions by supposing the law of the motion to be similar to that of a cycloidal pendulum, and as two of the waves cease to be transmitted in the regular way, we must suppose in the upper medium

$$\left.\begin{aligned} \psi &= \alpha \sin(ax+by+ct+e)+\beta \sin(-ax+by+ct+e_{,}) \\ \text{and} \quad \phi &= \varepsilon^{a'x}(A \sin\psi_0 + B\cos\psi_0) \end{aligned}\right\} \quad (20)$$

and in the lower one

$$\left.\begin{aligned} \psi_{,} &= \alpha_{,} \sin(a_{,}x+by+ct) \\ \phi_{,} &= \varepsilon^{-a_{,}'x}(A_{,} \sin\psi_0 + B_{,}\cos\psi_0) \end{aligned}\right\} \quad (21)$$

where to abridge $\psi_0 = by+ct$.

These substituted in the general equations (14) and (15), give

$$c^2 = \gamma^2(a^2+b^2) = \gamma_{,}^2(a_{,}^2+b^2) = g^2(-a'^2+b^2) = g_{,}^2(-a_{,}'^2+b^2),$$

or, since g and $g_{,}$ are both infinite,

$$b = a' = a_{,}'.$$

It only remains to substitute the values (20), (21) in the equations (17), which belong to the surface of junction, and thus we get

$$bA \sin\psi_0 + bB\cos\psi_0 + b\alpha\cos(\psi_0+e) + b\beta\cos(\psi_0+e_{,})$$
$$= -bA_{,}\sin\psi_0 - bB_{,}\cos\psi_0 + b\alpha_{,}\cos\psi_0,$$
$$bA\cos\psi_0 - bB\sin\psi_0 - a\alpha\cos(\psi_0+e) + a\beta\cos(\psi_0+e_{,})$$
$$= bA_{,}\cos\psi_0 - bB_{,}\sin\psi_0 - a_{,}\alpha_{,}\cos\psi_0. \quad (22)$$
$$\frac{1}{g^2}(A\sin\psi_0+B\cos\psi_0) = \frac{1}{g_{,}^2}(A_{,}\sin\psi_0+B_{,}\cos\psi_0),$$
$$\frac{1}{\gamma^2}\{\alpha\sin(\psi_0+e)+\beta\sin(\psi_0+e_{,})\} = \frac{1}{\gamma_{,}^2}\alpha_{,}\sin\psi_0.$$

Expanding the two last equations, comparing separately the coefficients of $\cos\psi_0$ and $\sin\psi_0$, and observing that

$$\frac{g}{g_{,}} = \frac{\gamma}{\gamma_{,}} = \mu \text{ suppose},$$

we get

$$\left.\begin{aligned} A &= \mu^2 A_{,}, \\ B &= \mu^2 B_{,}, \\ \alpha\cos e+\beta\cos e_{,} &= \mu^2\alpha_{,}, \\ \alpha\sin e+\beta\sin e_{,} &= 0 \end{aligned}\right\}. \quad (23)$$

In like manner the two first equations of (22) will give

$$0 = A+A_{,}-\alpha \sin e-\beta \sin e_{,},$$

$$0 = A-A_{,}+\frac{a_{,}\alpha_{,}}{b}+\frac{a}{b}(\beta \cos e_{,}-\alpha \cos e),$$

$$0 = B+B_{,}+\alpha \cos e+\beta \cos e_{,}-\alpha_{,},$$

$$0 = B-B_{,}+\frac{a}{b}(\beta \sin e_{,}-\alpha \sin e_{,});$$

combining these with the system (23), there results

$$\left.\begin{aligned} 0 &= A+A_{,} \\ 0 &= B+B_{,}+(\mu^2-1)\alpha_{,} \\ 0 &= A-A_{,}+\frac{a_{,}\alpha_{,}}{b}+\frac{a}{b}(\beta \cos e_{,}-\alpha \cos e) \\ 0 &= B-B_{,}+\frac{a}{b}(\beta \sin e_{,}-\alpha \sin e) \end{aligned}\right\}. \tag{24}$$

Again, the systems (23) and (24) readily give

$$\left.\begin{aligned} \alpha \sin e &= -\frac{1}{2}\cdot\frac{(\mu^2-1)^2}{\mu^2+1}\,\frac{b}{a}\alpha_{,} \\ \alpha \cos e &= \frac{1}{2}\cdot\left(\mu^2+\frac{a_{,}}{a}\right)\alpha_{,} \\ \beta \sin e_{,} &= \frac{1}{2}\cdot\frac{(\mu^2-1)^2}{\mu^2+1}\,\frac{b}{a}\alpha_{,} \\ \beta \cos e_{,} &= \frac{1}{2}\cdot\left(\mu^2-\frac{a_{,}}{a}\right)\alpha_{,} \end{aligned}\right\}; \tag{25}$$

and therefore

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2+1)^2\cdot\left(\mu^2-\frac{a_{,}}{a}\right)^2+(\mu^2-1)^4\frac{b^2}{a^2}}{(\mu^2+1)^2\cdot\left(\mu^2+\frac{a_{,}}{a}\right)^2+(\mu^2-1)^4\frac{b^2}{a^2}}. \tag{26}$$

When the refractive power in passing from the upper to the lower medium is not very great, μ does not differ much from 1. Hence, $\sin e$ and $\sin e_{,}$ are small, and $\cos e$, $\cos e_{,}$ do not differ sensibly from unity; we have, therefore, as a first approximation,

$$\frac{\beta}{\alpha} = \frac{\mu^2-\frac{a_{,}}{a}}{\mu^2+\frac{a_{,}}{a}} = \frac{\frac{\sin^2\theta}{\sin^2\theta_{,}}-\frac{\cot\theta_{,}}{\cot\theta}}{\frac{\sin^2\theta}{\sin^2\theta_{,}}+\frac{\cot\theta_{,}}{\cot\theta}} = \frac{\sin 2\theta-\sin 2\theta_{,}}{\sin 2\theta+\sin 2\theta_{,}} = \frac{\tan(\theta-\theta_{,})}{\tan(\theta+\theta_{,})},$$

which agrees with the formula in Airy's *Tracts*, p. 358†, for light polarized perpendicular to the plane of reflexion. This result is only a near approximation: but the formula (26) gives the correct value of B^2/α^2, or the ratio of the intensity of the reflected to the incident light; supposing, with all optical writers, that the intensity of light is properly measured by the square of the actual velocity of the molecules of the luminiferous ether.

From the rigorous value (26), we see that the intensity of the reflected light never becomes absolutely null, but attains a minimum value nearly when

$$0 = \mu^2-\frac{a_{,}}{a}, \quad \text{i.e., when} \quad \tan(\theta+\theta_{,}) = \infty, \tag{27}$$

which agrees with experiment, and this minimum value is, since (27) gives $b/a = \mu$,

$$\frac{\beta^2}{\alpha^2} = \frac{(\mu^2-1)^4\frac{b^2}{a^2}}{4(\mu^2+1)^4\,\mu^4+(\mu^2-1)^4\frac{b^2}{a^2}} = \frac{(\mu^2-1)^4}{4\mu^2(\mu^2+1)^2+(\mu^2-1)^4}\ \ldots \tag{28}$$

If $\mu = \frac{4}{3}$, as when the two media are air and water, we get

$$\frac{\beta^2}{\alpha^2} = \frac{1}{151} \text{ nearly.}$$

† [Airy, *ubi sup.* p. 110.]

It is evident from the formula (28), that the magnitude of this minimum value increases very rapidly as the index of refraction increases, so that for highly refracting substances, the intensity of the light reflected at the polarizing angle becomes very sensible, agreeably to what has been long since observed by experimental philosophers. Moreover, an inspection of the equations (25) will shew, that when we gradually increase the angle of incidence so as to pass through the polarizing angle, the change which takes place in the reflected wave is not due to an alteration of the sign of the coefficient β, but to a change of phase in the wave, which for ordinary refracting substances is very nearly equal to 180°; the minimum value of β being so small as to cause the reflected wave sensibly to disappear. But in strongly refracting substances like diamond, the coefficient β remains so large that the reflected wave does not seem to vanish, and the change of phase is considerably less than 180°. These results of our theory appear to agree with the observations of Professor Airy. (*Camb. Phil. Trans.* Vol. IV. p. 418, &c.)

Lastly, if the velocity $\gamma_{,}$ of transmission of a wave in the lower exceed γ that in the upper medium, we may, by sufficiently augmenting the angle of incidence, cause the refracted wave to disappear, and the change of phase thus produced in the reflected wave may readily be found. As the calculation is extremely easy after what precedes, it seems sufficient to give the result. Let therefore, here, $\mu = \gamma_{,}/\gamma$, also e, $e_{,}$ and θ as before, then $e_{,} = -e$, and the accurate value of e is given by

$$\tan e = \mu\sqrt{(\mu^2 \tan^2\theta - \sec^2\theta)} - \frac{(\mu^2-1)^2 \tan\theta}{\mu^2+1}.$$

The first term of this expression agrees with the formula of page 362, Airy's *Tracts*†, and the second will be scarcely sensible except for highly refracting substances.

† [Airy, *ubi sup.* p. 114, Art. 133.]

5. AN ESSAY TOWARDS A DYNAMICAL THEORY OF CRYSTALLINE REFLEXION AND REFRACTION*

J. MacCullagh

SECT. I. INTRODUCTORY OBSERVATIONS. EQUATION OF MOTION

Nearly three years ago I communicated to this Academy† the laws by which the vibrations of light appear to be governed in their reflexion and refraction at the surfaces of crystals. These laws—remarkable for their simplicity and elegance, as well as for their agreement with exact experiments—I obtained from a system of hypotheses which were opposed, in some respects, to notions previously received, and were not bound together by any known principles of mechanics, the only evidence of their truth being the truth of the results to which they led. On that occasion, however, I observed that the hypotheses were not independent of each other; and soon afterwards I proved that the laws of reflexion at the surface of a crystal are connected, in a very singular way, with the laws of double refraction, or of propagation in its interior; from which I was led to infer that "all these laws and hypotheses have a common source in other and more intimate laws which remain to

* *Transactions of the Royal Irish Academy*, **21** (1848). Read 9 December 1839.

† In a Paper "On the Laws of Crystalline Reflexion and Refraction."—*Transactions of the Royal Irish Academy*, Vol. xviii. p. 31. (*Supra*, p. 87.)

be discovered"; and that "the next step in physical optics would probably lead to those higher and more elementary principles by which the laws of reflexion and the laws of propagation are linked together as parts of the same system".† This step has since been made, and these anticipations have been realised. In the present Paper I propose to supply the link between the two sets of laws by means of a very simple theory, depending on certain special assumptions, and employing the usual methods of analytical dynamics.

In this theory, the two kinds of laws, being traced from a common origin, are at once connected with each other and severally explained; and it may be observed, that the explanation of each, as well as the source of their connexion, is now made known for the first time. For though the laws of crystalline propagation have attracted much attention during the period which has elapsed since they were discovered by Fresnel,‡ they have hitherto resisted every attempt that has been made to account for them by dynamical reasonings; and the laws of reflexion, when recently discovered, were apparently still more difficult to reach by such considerations. Nothing can be easier, however, than the process by which both systems of laws are now deduced from the same principles.

The assumptions on which the theory rests are these:—*First*, that the density of the luminiferous ether is a constant quantity; in which it is implied that this density is unchanged either by the motions which produce light or by the presence of material particles, so that it is the same within all bodies as in free space, and remains the same during the most intense vibrations. *Second*, that the vibrations in a plane-wave are rectilinear, and that, while the plane of the wave moves parallel to itself, the vibrations continue parallel to a fixed right line, the direction of this right line and the

† *Ibid*, p. 53, note. (*Supra*, p. 112.) The note here referred to was added some time after the Paper itself was read.

‡ These laws were published in his Memoir on Double Refraction—*Mémoires de l'Institut*, tom. vii. p. 45.

direction of a normal to the wave being functions of each other. This supposition holds in all known crystals, except quartz, in which the vibrations are elliptical.

Concerning the peculiar constitution of the ether we know nothing, and shall suppose nothing, except what is involved in the foregoing assumptions. But with respect to its physical condition generally, we shall admit, as is most natural, that a vast number of ethereal particles are contained in the differential element of volume; and, for the present, we shall consider the mutual action of these particles to be sensible only at distances which are insensible when compared with the length of a wave.

By putting together the assumptions we have made, it will appear that when a system of plane waves disturbs the ether, the vibrations are transversal, or parallel to the plane of the waves. For all the particles situated in a plane parallel to the waves are displaced, from their positions of rest, through equal spaces in parallel directions; and therefore if we conceive a closed surface of any form, including any volume great or small, to be described in the quiescent ether, and then all its points to partake of the motion imparted by the waves, any slice cut out of that volume, by a pair of planes parallel to the wave-plane and indefinitely near each other, can have nothing but its thickness altered by the displacements; and since the assumed preservation of density requires that the volume of the slice should not be altered, nor consequently its thickness, it follows that the displacements must be in the plane of the slice, that is to say, they must be parallel to the wave-plane. And conversely, when this condition is fulfilled, it is obvious that the entire volume, bounded by the arbitrary surface above described, will remain constant during the motion, while the surface itself will always contain within it the very same ethereal particles which it enclosed in the state of rest; and all this will be accurately true, no matter how great may be the magnitude of the displacements.

Let x, y, z be the rectangular co-ordinates of a particle before it is disturbed, and $x+\xi$, $y+\eta$, $z+\zeta$ its co-ordinates at the time t, the displacements ξ, η, ζ being functions of x, y, z and t. Let the ethereal density, which is the same in all media, be regarded as unity, so that $dxdydz$ may, at any instant, represent indifferently either the element of volume or of mass. Then the equation of motion will be of the form

$$\iiint dx\,dy\,dz\left(\frac{d^2\xi}{dt^2}\,\delta\xi+\frac{d^2\eta}{dt^2}\,\delta\eta+\frac{d^2\zeta}{dt^2}\,\delta\zeta\right)=\iiint dx\,dy\,dz\,\delta V, \tag{1}$$

where V is some function depending on the mutual actions of the particles. The integrals are to be extended over the whole volume of the vibrating medium, or over all the media, if there be more than one.

Setting out from this equation, which is the general formula of dynamics applied to the case that we are considering, we perceive that our chief difficulty will consist in the right determination of the function V; for if that function were known, little more would be necessary, in order to arrive at all the laws which we are in search of, than to follow the rules of analytical mechanics, as they have been given by Lagrange. The determination of V will, of course, depend on the assumptions above stated respecting the nature of the ethereal vibrations; but, before we proceed further, it seems advisable to introduce certain lemmas, for the purpose of abridging this and the subsequent investigations.†

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SECT. III. DETERMINATION OF THE FUNCTION ON WHICH THE MOTION DEPENDS. PRINCIPAL AXES OF A CRYSTAL

We come now to investigate the particular form which must be assigned to the function V, in order that the formula (1) may represent the motions of the ethereal medium. For this purpose conceive the plane of x' y' to be parallel to a system of plane waves whose vibrations are entirely transversal and parallel to the axis of y', so that $\xi'=0$, $\zeta'=0$. Imagine an elementary parallelpiped $dx'\,dy'\,dz'$, having its edges parallel to the axes of x', y', z', to be described in the ether when at rest, and then all its points to move according to the same law as the ethereal particles which compose it. The faces of the parallelpiped which are perpendicular to the edge dz' will be shifted, each in its own plane, in a direction parallel to the axis of y'; but their displacements will be unequal, and will differ by $d\eta'$, so that the edges connecting their corresponding angles will no longer be parallel to the axis of z', but will be inclined to it at an angle $\varkappa$ whose tangent is $d\eta'/dz'$.

Now the function V can only depend upon the directions of the axes of x', y', z' with respect to fixed lines in the crystal, and upon the angle $\varkappa$, which measures the change of form produced in the parallelpiped by vibration. This is the most general supposition which can be made concerning it. Since, however, by our second assumption, any one of these directions, suppose that of x', determines the other two, we may regard V as depending on the angle $\varkappa$ and on the direction of the axis of x' alone. But from the equations (F) it is manifest that the angle $\varkappa$ and the angles which the axis of x' makes with the fixed axes of x, y, z are all known when the quantities X, Y, Z are known. Consequently V is a function of X, Y, Z.

Supposing the angle $\varkappa$ to be very small, the quantities X, Y, Z will also be very small; and if V be expanded according to the powers of these quantities, we shall have

$$V = K+AX+BY+CZ+DX^2+EY^2+FZ^2$$
$$+GYZ+HXZ+IXY+\&c.,$$

the quantities K, A, B, C, D, &c., being constant. But in the state of equilibrium the value of δV ought to be nothing, in whatever way the position of the system be varied; that is to say, when the

† [Section II is here omitted.]

displacements ξ, η, ζ, and consequently the quantities X, Y, Z, are supposed to vanish, the quantity

$$\delta V = A\delta X + B\delta Y + C\delta Z + 2DX\delta X + \&c.,$$

ought also to vanish independently of the variations $\delta\xi$, $\delta\eta$, $\delta\zeta$, or, which comes to the same thing, independently of δX, δY, δZ. Hence† we must have $A = 0$, $B = 0$, $C = 0$; and therefore, if we neglect terms of the third and higher dimensions, we may conclude that the variable part of V is a homogeneous function of the second degree, containing, in its general form, the squares and products of X, Y, Z, with six constant coefficients.

Of these coefficients, the three which multiply the products of the variables may always be made to vanish by changing the directions of the axes of x, y, z. For this is a known property of functions of the second degree, when the co-ordinates are the variables; and we have shown, in Lemma II., that the quantities X, Y, Z are transformed by the very same relations as the co-ordinates themselves. Therefore, in every crystal there exist three rectangular axes, with respect to which the function V contains only the squares of X, Y, Z; and as it will presently appear that the coefficients of the squares must all be negative, in order that the velocity of propagation may never become imaginary, we may consequently write, with reference to these axes,

$$V = -\tfrac{1}{2}(a^2X^2 + b^2Y^2 + c^2Z^2), \tag{2}$$

omitting the constant K as having no effect upon the motion.

The axes of co-ordinates, in this position, are the *principal axes* of the crystal, and are commonly known by the name of *axes of elasticity*. Thus the existence of these axes is proved without any hypothesis respecting the arrangement of the particles of the medium. The constants a, b, c are the three principal velocities of propagation, as we shall see in the next section.

† See the reasoning of Lagrange in an analogous case *Mécanique Analytique*, tom. i. p. 68.

Having arrived at the value of V, we may now take it for the starting point of our theory, and dismiss the assumptions by which we were conducted to it. Supposing, therefore, in the first place, that a plane wave passes through a crystal, we shall seek the laws of its motion from equations (1) and (2), which contain everything that is necessary for the solution of the problem. The laws of propagation, as they are called, will in this way be deduced, and they will be found to agree exactly, so far as *magnitudes* are concerned, with those discovered by Fresnel; but the *direction* of the vibrations in a polarized ray will be different from that assigned by him. In the second place, we shall investigate the conditions which are fulfilled when light passes out of one medium into another, and we shall thus obtain the laws of reflexion and refraction at the surface of a crystal.

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6. ON A GYROSTATIC ADYNAMIC CONSTITUTION FOR 'ETHER'*

W. Thomson (Lord Kelvin)

1. Consider the double assemblage of the red and blue atoms of § 69 of Art. xcvii. above.† Annul all the forces of attraction and of repulsion between the atoms. Join each red to its blue neighbour by a rigid bar, as in the little model which I submitted to the Academy in my last communication. We shall thus have, abutting on each red atom and on each blue, four bars making between them obtuse angles, each equal to $\pi - \cos^{-1}\frac{1}{3}$.

2. Let us suppose that each atom be a little sphere, instead of being a point; that each bar is provided at its extremities with spherical caps (as in § 70 of Art. xcvii.), rigidly fixed to it, and kept in contact with the surface of the spheres by proper guards, leaving the caps free to slide upon the spherical surfaces. We shall thus have realised an articulated molecular structure, which in aggregate constitutes a perfect incompressible quasi-liquid. The deformations must be infinitely small, and such deformations imply diminutions of volume, infinitely small and of the second order, or proportional to their squares, which we may neglect. It is because of this limitation that we have not a perfect incompressible liquid, without the qualification "quasi". But this limitation does not alter at all the perfection of our ether, so far as concerns its fitness to transmit luminous waves.

* §§ 1–6 translated from *Comptes Rendus*, 16 Sept., 1889. §§ 7–15 from *Proceedings Royal Society of Edinburgh*, 17 Mar., 1890.

† [Cf. Appendix to this sele ction forrelevant sections of Art. xcvii. K.F.S.]

3. Now to give to our structure the quasi-elasticity which it requires in order to produce the luminous waves, let us attach to each bar a gyrostatic pair composed of two Foucault gyroscopes, mounted according to the following instructions.

4. Instead of a simple bar, let us take a bar of which the central part, for a third of its length for example, is composed of two rings in planes perpendicular to one another. Let the centre of each ring, and a diameter of each ring, be in the line of the bar. Let the two rings be the exterior rings of gyroscopes, and let the axes of the interior rings be mounted perpendicularly to the line of the bar. Let us now place the interior rings, with their planes in those of the exterior rings, and consequently with the axes of their flywheels in the line of the bar. Let us give speeds of rotation, equal, but in opposite directions, to the two flywheels.

5. The gyrostatic pair thus constituted (that is to say, thus constructed and thus energised) has the singular property of requiring a Poinsot couple to be applied to the bar in order to hold it at rest in any position inclined to the position in which it was given. The moment of this couple, L, remains sensibly constant until the axes of the flywheels have turned through considerable angles from their original direction in the primitive line of the bar; and is given by the following formula which is easily demonstrated by the theory of the gyroscope,

$$L = \frac{(mk^2\omega)^2}{\mu}\, i,$$

i meaning the angle, supposed infinitely small, between the length of the bar in its deviated position and in its primitive position, m meaning the mass of one of the flywheels, mk^2 meaning its moment of inertia, ω meaning its angular velocity, μ meaning the moment of inertia about the axis of the pivots of the interior ring, of the entire mass (ring and flywheel) which they support.

6. Our jointed structure, with the bars placed between the black and white atoms,† carrying the gyrostatic pairs, is not now as

† [Actually red and blue atoms – K.F.S.]

formerly without rigidity; but it has an altogether peculiar rigidity, which is not like that of ordinary elastic solids, of which the forces of elasticity depend simply on the deformations which they suffer. On the contrary, its forces depend directly on the absolute rotations of the bars and only depend on the deformations, because these are kinematic consequences of the rotations of the bars. This relation of the quasi-elastic forces with absolute rotation, is just that which we require for the ether, and especially to explain the phenomena of electro-dynamics and magnetism.

7. The structure thus constituted, though it has some interest as showing a special kind of quasi-solid elasticity, due to rotation of matter having no other properties but rigidity and inertia, does not fulfil exactly the conditions of Art. xcix., § 14. The irrotational distortion of the substance or structure, regarded as a homogeneous assemblage of double points, involves essentially rotations of some of the connecting bars, and therefore requires a balancing forcive. For the 'ether' of Art. xciv. no forcive must be needed to produce any irrotational deformation: and any displacement whether merely rotational, or rotational and deformational, must require a constant couple in simple proportion to the rotation and round the same axis. In a communication to the Royal Society of Edinburgh of a year ago, I stated the problem of constructing a jointed model under gyrostatic domination to fulfil the condition of having no rigidity against irrotational deformations, and of resisting rotation, or rotational deformation, with quasi-elastic forcive in simple proportion to rotation. I gave a solution, illustrated by a model, for the case of points all in one plane; but I did not then see any very simple three-dimensional solution. After many unavailing efforts, I have recently found the following.

8. Take six fine straight rods and six straight tubes all of the same length, the internal diameter of the tubes exactly equal to the external diameter of the rods. Join all the twelve together with ends to one point P. Mechanically this might be done (but it would

not be worth the doing), by a ball-and-twelve-socket mechanism. The condition to be fulfilled is simply that the axes of the six rods and of the six tubes all pass through one point P. Make a vast number of such clusters of six tubes and six rods, and, to begin with, place their jointed ends so as to constitute an equilateral homogeneous assemblage of points P, P', . . . each connected to its twelve nearest neighbours by a rod of one sliding into a tube of the other. This assemblage of points we shall call our primary assemblage. The mechanical connections between them do not impose any constraint: each point of the assemblage may be moved arbitrarily in any direction, while all the others are at rest. The mechanical connections exist merely for the sake of providing us with rigid lines joining the points, or more properly rigid cylindric surfaces having their axes in the joining lines. Make now a rigid frame G of three rods fixed together at right angles to one another through one point O. Place it with its three bars in contact with the three pairs of rigid sides of any tetrahedron

$$(PP', P''P'''), \quad (PP'', P'''P'), \quad (PP''', P''P'),$$

of our primary assemblage. Place similarly other similar rigid frames G, G', &c., on the edges of all the tetrahedrons congener (Art. xcvii., § 13) to the one first chosen, the points O, O', $O'O''$ &c. form a second homogeneous assemblage, related to the assemblage of *P's* just as the reds are related to the blues in Art. xcvii., § 69.

9. The position of the frame G, that is to say its orientation and the position of its centre O (six disposables) is completely determined by the four points P, P', P'', P'''. (Thomson and Tait's *Natural Philosophy*, § 198, and *Elements*, § 168.) If its bars were allowed to break away from contact with the three pairs of edges of the tetrahedrons, we might choose as its six coordinates, the six distances of its three bars from the three pairs of edges; but we suppose it to be constrained to preserve these contacts. And now let any one of the points P, P', P'', P''' or all of them be moved

in any manner, the position of the frame G is always fully determinate. This is illustrated by a model accompanying the present communication, showing a single tetrahedron of the primary assemblage and a single G frame. The edges of the tetrahedron are of copper wires sliding into glass tubes. The wires and tubes are provided with an eye or staple respectively, through which a ring passes to hold three ends together at the corners. Two of the rings have two glass tubes and one copper wire linked on each, while the other two rings have each two copper wires and one glass tube.

10. Returning now to our multitudinous assemblage, let it be displaced by stretchings of all the edges parallel to PP' with no rotation of PP' or $P''P'''$. This constitutes a homogeneous irrotational deformation of the primary assemblage. The frames G, G', &c. experience merely translatory motions without any rotation, as we see readily by confining our attention to G and the tetrahedron PP', $P''P'''$. Consider similarly five other displacements by stretchings parallel to the five other edges of the tetrahedron. Any infinitely small homogeneous deformation of the primary assemblage (§ 8 above), may be determinately resolved into six such simple stretchings, and any infinitely small rotational deformations may be produced by the superposition of a rotation without deformation, upon the irrotational deformation. Hence an infinitely small homogeneous deformation of the primary assemblage without rotation, produces only translatory motion, no rotation of the G frames: and any infinitely small homogeneous displacement whatever of the primary assemblage, produces a rotation of each frame equal to, and round the same axis as, its own rotational component.

11. It now only remains to give irrotational stability to the G frames. This may be done by mounting gyrostats properly upon them according to the principle stated in §§ 3–5 above and Art. cii. §§ 21–26 below. Three gyrostats would suffice but twelve may be taken for symmetry and for avoidance of any resultant mo-

ment of momentum of all the rotators mounted on one frame. Instead of ordinary gyrostats with rigid flywheels we may take liquid gyrostats as described below, § 12, and so make one very small step towards abolishing the crude mechanism of flywheels and axles and oiled pivots. But I chose the liquid gyrostat at present merely because it is more easily described.

12. Imagine a hollow anchor ring, or tore, that is to say an endless circular tube of circular cross-section. Perforate it in the line of a diameter and fix into it tubes to guard the perforations as shown in the accompanying diagram. Fill it with frictionless liquid, and give the liquid irrotational circulatory motion as indicated by the arrow heads in the diagram. This arrangement constitutes the hydrokinetic substitute for our mechanical flywheel. Mount it on a stiff diametral rod passing through the perforations,

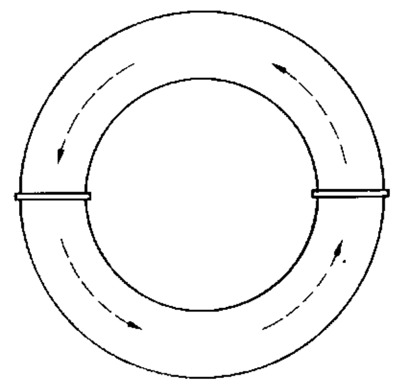

FIG. 6.1

and it becomes the mounted gyrostat, or Foucault gyroscope, required for our model. Looking back to §§ 3–4 above we see how much its use would have simplified and shortened the descriptions there given, which however was given purposely as they were because they describe real mechanism by which the exigences of our model can be practically realised in a very interesting and instructive manner, as may be seen in Art. cii., §§ 21–23 below.

13. Let XOX', YOY', ZOZ' be the three bars of the G frame: mount upon each of them four of our liquid gyrostats, those on XOX' being placed as follows and the others correspondingly. Of the four rings mounted on XX' two are to be placed in the plane of YY', XX', the other two in the plane of ZZ', XX'. The circuital fluid motions are to be in opposite directions in each pair.

14. The gyrostatic principle stated in § 5 above, applied to our G frame, with the twelve liquid gyrostats thus mounted on it, shows that if, from the position in which it was given with all the rings at rest, it be turned through an infinitesimal angle i round any axis, it requires, in order to hold it at rest in this altered position, a couple in simple proportion to i; and that this couple remains sensibly constant, as long as the planes of all the gyrostats have only changed by very small angles from parallelism to their original directions. Hence with this limitation as to time our primary homogeneous assemblage of points controlled by the gyrostatically dominated frames G, G' &c. fulfils exactly the condition stated for the ideal ether of § 14 of Art. xcix. If the velocity of the motion of the liquid in each gyrostat be infinitely great, the system exerts infinite resistance against rotation round any axis; and if the bars and tubes constituting the edges of the tetrahedron, and the bars of the G frames are all perfectly rigid, the primary assemblage is incapable of rotation or of rotational deformation: but if there is some degree of elastic flexural yielding in the edges of the tetrahedron, or in the bars of the G frame, or in all of them, the primary assemblage fulfils the definition of gyrostatic rigidity of § 14 Art. xciv. without any limit as to time, that is to say with perfect durability of its quasi-elastic rigidity.

15. A homogeneous assemblage of points with gyrostatic quasi rigidity conferred upon it in the manner described in §§ 8–14 would, if constructed on a sufficiently small scale, transmit vibrations of light exactly as does the ether of nature: and it would be incapable

of transmitting condensational-rarefactional waves, because it is absolutely devoid of resistance to condensation and rarefaction. It is in fact, a mechanical realisation of the medium to which I was led one and a half years ago,† from Green's original theory, by purely optical reasons, in endeavouring to explain results of observation regarding the refraction and reflection of light.

APPENDIX

Being article XCVII, §§ 66–70, of W. Thomson's Math. and Phys. Papers, iii. 425 ff.

§ 66. Try first to realise an incompressible elastic solid. When this is done we shall see, by an inevitably obvious modification, how to give any degree of compressibility we please without changing the rigidity, and so to realise an elastic solid with any given positive rigidity, and any given positive or negative bulk-modulus (stable without any surface constraint, only when the bulk-modulus is positive).

§ 67. To aid conception, make a tetrahedronal model of six equal straight rods, jointed at the angular points in which three meet, each having longitudinal elasticity with perfect anti-flexural rigidity. These constitute merely an ideal materialisation of the connection assumed in the Boscovich attractions and repulsions. A very telling *realisation* of the system thus imagined is made by taking six equal and similar bent bows and jointing their ends together by threes. The jointing might be done accurately by a ball and double socket mechanism of an obvious kind, but it would not be worth the doing. A rough arrangement of six bows of bent steel wire, merely linked together by hooking an end of one into rings on the ends of two others, may be made in a few minutes; and even its defects are not unhelpful towards a vivid understanding of our subject. We have now an element of elastic

† *Philosophical Magazine*, Nov. 1888, On the reflection and refraction of light, by Sir W. Thomson.

solid which clearly has an essentially definite ratio of compressibility to reciprocal of either of the rigidities (§ 27 above), each being inversely proportional to the stiffness of the bows. Now we can obviously make this solid incompressible if we take a boss jointed to four equal tie-struts, and joint their free ends to the four corners of the tetrahedron; and we do not alter either of the rigidities if the length of each tie-strut is equal to distance from centre to corners of the unstressed tetrahedron. If the tie-struts are shorter than this, their effect is clearly to augment the rigidities; if longer, to diminish the rigidities. The mathematical investigation proves that it diminishes the greater of the rigidities more than it diminishes the less, and that before it annuls the less it equalises the greater to it.

§ 68. If for the present we confine our attention to the case of the tie-struts longer than the un-strained distance from centre to corners, simple struts will serve; springs, such as bent bows, capable of giving thrust as well as pull along the sides of the tetrahedron, are not needed; mere india-rubber elastic filaments will serve instead, or ordinary spiral springs, and all the end-jointings become much simplified. A realised model accompanies this communication.

§ 69. The model being completed, we have two simple homogeneous Bravais assemblages of points; reds and blues, as we shall call them for brevity; so placed that each blue is in the centre of a tetrahedron of reds, and each red in the centre of a tetrahedron of blues. The other tetrahedronal groupings (Molecular Tactics, §§ 45, 60) being considered, each tetrahedron of reds is vacant of blue, and each tetrahedron of blues is vacant of reds.†

† An interesting structure is suggested by adding another homogeneous assemblage, marked green; giving a green in the centre of each hitherto vacant tetrahedron of reds. It is the same assemblage of triplets as that described in § 24 above. It does not (as long as we have mere jointed struts of constant length between the greens and reds) modify our rigidity-modulus, nor otherwise help us at present, so, having inevitably noticed it, we leave it.

§ 70. Imagine the springs removed and the struts left; but now all properly jointed by fours of ends with perfect frictionless ball-and-socket triple-joints. We have a perfectly non-rigid three-dimensional skeleton frame-work, analogous to idealised plane netting consisting of stiff straight sides of hexagons perfectly jointed in threes of ends. [Compare Art. C., § 2, below.]

7. ON THE ELECTROMAGNETIC THEORY OF THE REFLECTION AND REFRACTION OF LIGHT*

G. F. FITZGERALD

IN THE second volume of his "Electricity and Magnetism", Professor J. Clerk Maxwell has proposed a very remarkable electromagnetic theory of light, and has worked out the results as far as the transmission of light through uniform crystalline and magnetic media are concerned, leaving the questions of reflection and refraction untouched. These, however, may be very conveniently studied from his point of view.

If we call W the electrostatic energy of the medium, it may be expressed in terms of the electromotive force and the electric displacement at each point, as is done in Professor Maxwell's "Electricity and Magnetism", vol. ii, part 4, ch. 9. I shall adopt his notation, and call the electromotive force $\mathfrak{E}$, and its components P, Q, R; and the electric displacement $\mathfrak{D}$, and its components f, g, h. As several of the results of this Paper admit of a very elegant expression in Quaternion notation, I shall give the work and results in both Cartesian and Quaternion form, confining the German letters to the Quaternion notation. Between these quantities, then,

* From the *Philosophical Transactions of the Royal Society* (Pt. II, 1880; Art. xix, p. 691). Communicated by G. J. Stoney, M.A., F.R.S., Secretary of the Queen's University in Ireland. Received 26 October, 1878. Read 9 January, 1879.

we have the equation

$$W = -\tfrac{1}{2}\iiint S\mathfrak{E}\mathfrak{D}\,dx\,dy\,dz = \tfrac{1}{2}\iiint (Pf+Qg+Rh)\,dx\,dy\,dz.$$

Similarly, the kinetic energy T may be expressed in terms of the magnetic induction $\mathfrak{B}$, and the magnetic force $\mathfrak{H}$, or their components a, b, c, and α, β, γ, by the equation

$$T = -\frac{1}{8\pi}\iiint S\mathfrak{B}\mathfrak{H}\,dx\,dy\,dz$$
$$= \frac{1}{8\pi}\iiint (a\alpha+b\beta+c\gamma)\,dx\,dy\,dz.$$

I shall at present assume this to be a complete expression for T, and return to the case of magnetized media for separate treatment, as Professor Maxwell has proposed additional terms in this case, in order to account for their property of rotatory polarization. I shall throughout assume the media to be isotropic as regards magnetic induction, for the contrary supposition would enormously complicate the question, and be, besides, of doubtful physical applicability. For the present I shall not assume them to be electrostatically isotropic. Hence $\mathfrak{E}$ is a linear vector and self-conjugate function of $\mathfrak{D}$, and consequently P, Q, R linear functions of f, g, h, so that we may write in Quaternion notation

$$\mathfrak{E} = \phi\mathfrak{D};$$

and if we call U the general symmetrical quadratic function of f, g, h, we may assume

$$U = Pf+Qg+Rh,$$

and consequently

$$W = -\tfrac{1}{2}\iiint S\mathfrak{D}\phi\mathfrak{D}\,dx\,dy\,dz = \tfrac{1}{2}\iiint U\,dx\,dy\,dz.$$

As the medium is magnetically isotropic, we have

$$\mathfrak{B} = \mu\mathfrak{H}, \quad \text{or} \quad a = \mu\alpha, \quad b = \mu\beta, \quad c = \mu\gamma,$$

where μ is the coefficient of magnetic inductive capacity, and consequently the electrokinetic energy may be written

$$T = -\frac{\mu}{8\pi}\iiint \mathfrak{H}^2\, dx\, dy\, dz = \frac{\mu}{8\pi}\iiint (\alpha^2+\beta^2+\gamma^2)\, dx\, dy\, dz.$$

Now I shall assume the mediums to be non-conductors; and although this limits to some extent the applicability of my results, and notably their relation to metallic reflection, yet it is a necessity, for otherwise the problem would be beyond my present powers of solution. With this assumption, and using Newton's notation of $\dot{x}$ for dx/dt, we have the following equations (see "Electricity and Magnetism", vol. ii., § 619):—

$$4\pi\dot{\mathfrak{D}} = V \nabla \mathfrak{H},$$

using ∇ for the operation

$$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz};$$

or the same in terms of its components, namely,

$$4\pi\dot{f} = \frac{d\gamma}{dy}-\frac{d\beta}{dz},$$

$$4\pi\dot{g} = \frac{d\alpha}{dz}-\frac{d\gamma}{dx},$$

$$4\pi\dot{h} = \frac{d\beta}{dx}-\frac{d\alpha}{dy}.$$

Assuming now a quantity $\mathfrak{R}$ with components ξ, η, ζ, such that

$$\mathfrak{R} = \int \mathfrak{H}\, dt,$$

and consequently

$$\dot{\mathfrak{R}} = \mathfrak{H};$$

or, in terms of the components,

$$\dot{\xi} = \alpha, \quad \dot{\eta} = \beta, \quad \dot{\zeta} = \gamma,$$

we may evidently write

$$4\pi\mathfrak{D} = V \nabla \mathfrak{R},$$

i.e., $\displaystyle 4\pi f = \frac{d\zeta}{dy}-\frac{d\eta}{dz}, \quad 4\pi g = \frac{d\xi}{dz}-\frac{d\zeta}{dx}, \quad 4\pi h = \frac{d\eta}{dx}-\frac{d\xi}{dy},$

so that we have

$$W = -\frac{1}{32\pi^2}\iiint S(V \nabla\mathfrak{R}\cdot\phi V \nabla\mathfrak{R})\, dx\, dy\, dz,$$

$$T = -\frac{\mu}{8\pi}\iiint \dot{\mathfrak{R}}^2\, dx\, dy\, dz = \frac{\mu}{8\pi}\iiint (\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2)\, dx\, dy\, dz.$$

Lagrange's equations of motion may often be very conveniently represented as the conditions that $\int(T-W)\,dt$ should be a minimum, or, in other words, that

$$\delta\int (T-W)\, dt = 0,$$

and this method, from its symmetry, is particularly applicable to the methods of Quaternions.

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8. THE ROTATIONAL ETHER IN ITS APPLICATION TO ELECTROMAGNETISM*

O. HEAVISIDE

ACCORDING to Maxwell's theory of electric displacement, disturbances in the electric displacement and magnetic induction are propagated in a non-conducting dielectric after the manner of motions in an incompressible solid. The subject is somewhat obscured in Maxwell's treatise by his equations of propagation containing A, Ψ, J, all of which are functions considerably remote from the vectors which represent the state of the field, viz., the electric and magnetic forces, and by some dubious reasoning concerning Ψ and J. There is, however, no doubt about the statement with which I commenced, as it becomes immediately evident when we ignore the potentials and use E or H instead, the electric or the magnetic force.

The analogy has been made use of in more ways than one, and can be used in very many ways. The easiest of all is to assume that the magnetic force is the velocity of the medium, magnetic induction the momentum, and so on, as is done by Prof. Lodge (Appendix to "Modern Views of Electricity"). I have also used this method for private purposes, on account of the facility with which electromagnetic problems may be made elastic-solid problems. I have shown that when impressed electric force acts it is the curl or rotation of the electric force which is to be considered

* O. Heaviside *Electromagnetic Theory*, Dover edit: New York, 1950. Originally published in the January, 1891 issue of *The Electrician*, **26,** 360.

as the source of the resulting disturbances. Now, on the assumption that the magnetic force is the velocity in the elastic solid, we find that the curl of the impressed electric force is represented simply by impressed mechanical force of the ordinary Newtonian type. This is very convenient.

But the difficulties in the way of a complete and satisfactory representation of electromagnetic phenomena, by an elastic-solid ether are insuperable. Recognising this, Sir W. Thomson has recently brought out a new ether; a rotational ether. It is incompressible, and has no true rigidity, but possesses a quasi-rigidity arising from elastic resistance to absolute rotation.

The stress consists partly of a hydrostatic pressure (which I shall ignore later), but there is no distorting stress, and its place is taken by a rotating stress. It gives rise to a translational force and a torque. If **E** be the torque, the stress on any plane N (unit normal) is simply **VEN**, the vector product of the torque and the normal vector.

The force is $-\text{curl}\,\mathbf{E}$. We have therefore the equation of motion

$$-\text{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}},$$

if H is the velocity and μ the density. But, alas, the torque is proportional to the rotation. This gives

$$\text{curl}\,\mathbf{H} = c\dot{\mathbf{E}},$$

where c is the compliancy, the reciprocal of the quasi-rigidity.

Now these are the equations connecting electric and magnetic force in a non-conducting dielectric, when μ is the inductivity and c the permittancy. We have a parallelism in detail, not merely in some particulars. The kinetic energy $\frac{1}{2}\mu H^2$ represents the magnetic energy, and the potential energy $\frac{1}{2}cE^2$ the electric energy. The vector-flux of energy is **VEH**, the activity of the stress.

This mode of representation differs from that of Sir W. Thomson, who represents magnetic force by rotation. This system makes electric energy kinetic, and magnetic energy potential, which I do not find so easy to follow.

Now let us, if possible, extend our analogy to conductors. Let the translational and the rotational motions be both frictionally resisted, and let the above equations become

$$-\text{curl}\,\mathbf{E} = g\mathbf{H}+\mu\dot{\mathbf{H}},$$

$$\text{curl}\,\mathbf{H} = k\mathbf{E}+c\dot{\mathbf{E}},$$

where g is the translational frictionality; k will be considered later. We have now the equations of electric and magnetic force in a dielectric with duplex conductivity, k being the electric and g the magnetic conductivity (by analogy with electric force, but a frictionality in our present dynamical analogy).

We have, therefore, still a parallelism in every detail. We have waste of energy by friction $g\mathrm{H}^2$ (translational) and $k\mathrm{E}^2$ (rotational). If $g/\mu = k/c$ the propagation of disturbances will take place precisely as in a non-conducting dielectric, though with attenuation caused by the loss of energy.

To show how this analogy works out in practice, consider a telegraph circuit, which is most simply taken to be three co-axial tubes. Let, A, B, and C be the tubes; A the innermost, C the outermost, B between them; all closely fitted. Let their material be the rotational ether. In the first place, suppose that there is perfect slip between B and its neighbours. Then, when a torque is applied to the end of B (the axis of torque to be that of the tubes), and circular motion thus given to B, the motion is (in virtue of the perfect slip) transmitted along B, without change of type, at constant speed, and without affecting A and C.

This is the analogue of a concentric cable, if the conductors A and C be perfect conductors, and the dielectric B a perfect insulator. The terminal torque corresponds to the impressed voltage. It should be so distributed over the end of B that the applied force there is circular tangential traction, varying inversely as the distance from the axis; like the distribution of magnetic force, in fact.

Now, if we introduce translational and rotational resistance in B, in the above manner, still keeping the slip perfect, we make the dielectric not only conducting electrically but also magnetically.

This will not do. Abolish the translational resistance in B altogether, and let there be no slip at all between B and A, and B and C. Let also there be rotational resistance in A and C.

We have now the analogue of a real cable: two conductors separated by a third. All are dielectrics, but the middle one should have practically very slight conductivity, so that it is pre-eminently a dielectric; whilst the other two should have very high conductivity, so that they are pre-eminently conductors. The three constants, μ, c, k, may have any value in the three tubes, but practically k should be in the middle tube a very small fraction of what it is in the others.

It is remarkable that the *quasi-rotational* resistance in A and C should tend to counteract the distorting effect on waves of the *quasi-rotational* resistance in B. But the two rotations, it should be observed, are practically perpendicular, being axial or longitudinal (now) in A and C, and transverse or radial in B; due to the relative smallness of k in the middle tube.

To make this neutralising property work exactly we must transfer the resistance in the tubes A and C to the tube B, at the same time making it translational resistance. Also restore the slip. Then we can have perfect annihilation of distortion in the propagation of disturbances, viz., when k and g are so proportioned as to make the two wastes of energy equal. In the passage of a disturbance along B there is partial absorption, but no reflection.

But as regards the meaning of the above k there is a difficulty. In the original rotational ether the torque varies as the rotation. If we superadd a real frictional resistance to rotation we get an equation of the form

$$\dot{\mathbf{E}} = \left(a+b\frac{d}{dt}\right)\text{curl}\,\mathbf{H},$$

E being (as before) the torque, and $\mathbf{H}$ the velocity. But this is not of the right form, which is (as above)

$$\text{curl}\,\mathbf{H} = \left(k+c\frac{d}{dt}\right)\mathbf{E};$$

therefore some special arrangement is required (to produce the dissipation of energy $k\mathrm{E}^2$), which does not obviously present itself in the mechanics of the rotational ether.

On the other hand, if we follow up the other system, in which magnetic force is allied with rotation, we may put $g = 0$, let $-\mathbf{E}$ be the velocity and $\mathbf{H}$ the torque; μ the compliancy, c the density, and k the translational frictionality. This gives

$$-\mathrm{curl}\,\mathbf{E} = \mu\dot{\mathbf{H}}$$

$$\mathrm{curl}\,\mathbf{H} = k\mathbf{E}+c\dot{\mathbf{E}}.$$

We thus represent a homogeneous conducting dielectric, with a translational resistance to cause the Joulean waste of energy. But it is now seemingly impossible to properly satisfy the conditions of continuity at the interface of different media. For instance, the velocity $-\mathbf{E}$ should be continuous, but we do not have normal continuity of electric force at an interface. In the case of the tubes we avoided this difficulty by having the velocity tangential.

Either way, then, the matter is left, for the present, in an imperfect state.

In the general case, the d/dt of our equations should receive an extended meaning, on account of the translational motion of the medium. The analogy will, therefore, work out less satisfactorily. And it must be remembered that it is only an analogy in virtue of similitude of relations. We cannot, for instance, deduce the Maxwellian stresses and mechanical forces on charged or currented bodies. The similitude does not extend so far. But certainly the new ether goes somewhat further than anything known to me that has been yet proposed in the way of a stressed solid.

[P.S.—The special reckonings of torque and rotation in the above are merely designed to facilitate the elastic-solid and electro-magnetic comparisons without unnecessary constants.]

9. AETHER AND MATTER*

J. LARMOR

DYNAMICAL THEORY OF ELECTRICAL ACTIONS

Least Action, fundamental in General Dynamics

49. THE idea of deducing all phenomenal changes from a principle of least expenditure of effort or action dates for modern times, as is well known, from the speculations of Maupertuis. The main illustration with which he fortified his view was Fermat's principle of least time for ray propagation in optics. This optical law follows as a direct corollary from Huygens' doctrine that radiation is propagated by wave-motions. In Maupertuis' hands, however, it reverted to the type of a dogma of least action in the dynamical sense as originally enunciated vaguely by Descartes, which Fermat's statement of the principle as one of least time was intended to supersede; under that aspect it was dynamically the equally immediate corollary of the corpuscular theory of optical rays which was finally adopted by Newton.

The general idea of Maupertuis at once attracted the attention of mathematicians; and the problem of the exact specification of the Action, so as to fulfil the minimum relation, was solved by Euler for the case of orbits of particles. Shortly afterwards the

* *Aether and Matter*, published by Cambridge University Press, Cambridge, 1900.

solution was re-stated with greater precision, and generalized to all material systems, by Lagrange (*Mem. Taurin.*, 1760) in one of his earliest and most brilliant memoirs, which constructed the algorithm of the Calculus of Variations, and at the same time also laid the foundation of the fundamental physical science of Analytical Dynamics. The subsequent extensions by Hamilton of the Lagrangian analytical procedure involve, so far as interpretation has hitherto been enabled to go, rather fundamental developments in the mathematical methods than new physical ideas,—except in the weighty result that the mere expression of all the quantities of the system as differential coefficients of a single characteristic function establishes relations of complete reciprocity between them, and also between the various stages, however far apart in time, of the system's progress.

It is now a well-tried resource to utilize the principle that every dynamical problem can be enunciated, in a single formula, as a variation problem, in order to help in the reduction to dynamics of physical theories in which the intimate dynamical machinery is more or less hidden from direct inspection. If the laws of any such department of physics can be formulated in a minimum or variational theorem, that subject is thereby virtually reduced to the dynamical type: and there remain only such interpretations, explanations, and developments, as will correlate the integral that is the subject of variation with the corresponding integrals relating to known dynamical systems. These developments will usually take the form of the tracing out of analogies between the physical system under consideration and dynamical systems which can be directly constructed to have Lagrangian functions of the same kind: they do not add anything logically to the completeness and sufficiency of the analytical specification of the system, but by being more intuitively grasped by the mind and of more familiar type, they often lead to further refinements and developments which carry on our theoretical views into still higher and more complete stages.

Derivation of the Equations of the Electric Field from the Principle of Least Action

50. It has been seen (§ 48) that the only effective method of working out the dynamics of molecular systems is to abolish the idea of force between the molecules, about which we can directly know nothing, and to formulate the problem as that of the determination of the natural sequence of changes of configuration in the system. If the individual molecules are to be permanent, the system, when treated from the molecular standpoint, must be conservative; so that the Principle of Least Action supplies a foundation certainly wide enough, if only it is not beyond our powers of development.

We require first to construct a dynamical scheme for the free aether when no material molecules are present. It is of course an elastic medium: let us assume that it is practically at rest, and let the vector (ξ, η, ζ) represent the displacement, elastic and other, of its substance at the point (x, y, z) which arises from the strain existing in it. We assume (to be hereafter verified by the results of the analysis) for its kinetic energy T and its potential energy W the expressions

$$T = \tfrac{1}{2}A \int (\dot\xi^2+\dot\eta^2+\dot\zeta^2)\, d\tau$$

$$W = \tfrac{1}{2}B \int (f^2+g^2+h^2)\, d\tau$$

in which $\delta\tau$ denotes an element of volume, A and B are constants, the former a constant of inertia, the latter a modulus of elasticity, and in which (f, g, h) is a vector defined as regards its mode of change† by the relation

$$(\dot f, \dot g, \dot h) = \frac{1}{4\pi}\left(\frac{d\dot\zeta}{dy}-\frac{d\dot\eta}{dz},\quad \frac{d\dot\xi}{dz}-\frac{d\dot\zeta}{dx},\quad \frac{d\dot\eta}{dx}-\frac{d\dot\xi}{dy}\right), \qquad \text{(I)}$$

† This allows for the permanent existence, independently of $(\dot\xi, \dot\eta, \dot\zeta)$, of the intrinsic aethereal displacement surrounding each electron. Cf. Appendix E.

where the 4π is inserted in order to conform to the ordinary electrical usage.

This definition makes

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0,$$

so that (f, g, h) is a stream vector.

To obtain the dynamical equations of this medium, we have to develop the variational equation

$$\delta\int(T-W)\,dt=0,$$

subject to the time of motion being unvaried.

Now

$$\begin{aligned}\delta\int T dt &= A\int dt\int(\dot\xi\delta\dot\xi+\dot\eta\delta\dot\eta+\dot\zeta\delta\dot\zeta)\,d\tau\\ &= A\left|\int(\dot\xi\delta\xi+\dot\eta\delta\eta+\dot\zeta\delta\zeta)\,d\tau\right|_{t_1}^{t_2}\\ &\quad -A\int dt\int(\ddot\xi\delta\xi+\ddot\eta\delta\eta+\ddot\zeta\delta\zeta)\,d\tau.\end{aligned}$$

Also

$$\begin{aligned}\delta W &= \frac{B}{4\pi}\int\left\{f\left(\frac{d\delta\zeta}{dy}-\frac{d\delta\eta}{dz}\right)+g\left(\frac{d\delta\xi}{dz}-\frac{d\delta\zeta}{dx}\right)+h\left(\frac{d\delta\eta}{dx}-\frac{d\delta\xi}{dy}\right)\right\}d\tau\\ &= \frac{B}{4\pi}\int\{(ng-mh)\,\delta\xi+(lh-nf)\,\delta\eta+(mf-lg)\,\delta\zeta\}\,dS\\ &\quad +\frac{B}{4\pi}\int\left\{\left(\frac{dh}{dy}-\frac{dg}{dz}\right)\delta\xi+\left(\frac{df}{dz}-\frac{dh}{dx}\right)\delta\eta+\left(\frac{dg}{dx}-\frac{df}{dy}\right)\delta\zeta\right\}d\tau\end{aligned}$$

where (l, m, n) is the direction vector of the element of boundary surface δS.

In these reductions by integration by parts the aim has been as usual to express dependent variations such as $\delta\dot\xi$, $d\delta\zeta/dy$, in terms

of the independent ones $\delta\xi$, $\delta\eta$, $\delta\zeta$. This requires the introduction of surface integrals: if the region under consideration is infinite space, and the exciting causes of the disturbance are all at finite distance from the origin, these surface integrals over an infinitely remote boundary cannot in the nature of things be of influence on the state of the system at a finite distance, and in fact it may be verified that they give a null result: in other cases they must of course be retained.

On substitution in the equation of Action of these expressions for the variations, the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$ must separately vanish both in the volume integral and in the surface integral, since $\delta\xi$, $\delta\eta$, $\delta\zeta$ are perfectly independent and arbitrary both at each element of volume $\delta\tau$ and at each element of surface δS. This gives, from the volume integral, the equations of vibration or wave-propagation

$$\frac{B}{4\pi}\left(\frac{dh}{dy}-\frac{dg}{dz},\ \frac{df}{dz}-\frac{dh}{dx},\ \frac{dg}{dx}-\frac{df}{dy}\right)=-A(\ddot\xi,\ddot\eta,\ddot\zeta). \qquad \text{(II)}$$

The systems of equations (I) and (II), thus arrived at, become identical in form with Maxwell's circuital equations which express the electrostatic and electrodynamic working of free aether, if $(\dot\xi, \dot\eta, \dot\zeta)$ represents the magnetic induction and (f, g, h) the aethereal displacement; the velocity of propagation is $(4\pi)^{-1}(B/A)^{\frac{1}{2}}$, so that $B/A = 16\pi^2c^2$ where c is the velocity of radiation. They are also identical with MacCullagh's optical equations, the investigation here given being in fact due to him.

51. Now let us extend the problem to aether containing a system of electrons or discrete electric charges. Each of these point-charges determines a field of electric force around it: electric force must involve aether-strain of some kind, as has already been explained: thus an electric point-charge is a nucleus of intrinsic strain in the aether. It is not at present necessary to determine what kind of permanent configuration of strain in the aether this can be, if only we are willing to admit that it can move or slip freely about through

that medium much in the way that a knot slips along a rope: we thus in fact treat an electron or point-charge of strength e as a freely mobile singular point in the specification of the aethereal strain (f, g, h), such that very near to it (f, g, h) assumes the form

$$-\frac{e}{4\pi}\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\frac{1}{r}.$$

We can avoid the absolutely infinite values, at the origin of the distance r, by treating the nucleus of the permanent strain-form not as a point but as a very minute region:† this analytical artifice will keep all the elements of the integrals of our analysis finite, while it will not affect any physical application which considers the electron simply as a local charge of electricity of definite amount.

Now provided there is nothing involved in the electron except a strain-form, no inertia or energy foreign to the aether residing in its nucleus such as would prevent free unresisted mobility, as it is perhaps difficult to see how there could be, the equations (I) and (II) still determine the state of the field of aether, at any instant, from its state, supposed completely known, at the previous instant: and this determination includes a knowledge of the displacement of the nucleus of each strain-form during the intervening element of time. These equations therefore suffice to trace the natural sequence of change in the complex medium thus constituted by the aether and the nuclei pervading it. But if the nuclei had inertia and mutual actions of their own, independent of the aether, there would in addition to the continuous equations of motion of the aether itself be dynamical equations of motion for each strain-form as well, which would interact and so have to be combined into continuity with the aethereal equations, and the problem would assume a much more complex form: in other words, the complete energy function employed in formulating the Principle of Least

† This substitution affects only the *intrinsic* molecular energy; cf. *Phil. Trans.* 1894 A, pp. 812–3.

Action would also involve these other types of physical action, if they existed.

52. But for purposes of the electrodynamic phenomena of material bodies, which we can only test by observation and experiment on matter in bulk, a complete atomic analysis of the kind thus indicated would (even if possible) be useless; for we are unable to take direct cognizance of a single molecule of matter, much less of the separate electrons in the molecule to which this analysis has regard. The development of the theory which is to be in line with experience must instead concern itself with an effective differential element of volume, containing a crowd of molecules numerous enough to be expressible continuously, as regards their average relations, as a volume-density of matter. As regards the actual distribution in the element of volume of the really discrete electrons, all that we can usually take cognizance of is an excess of one kind, positive or negative, which constitutes a volume density of electrification, or else an average polarization in the arrangement of the groups of electrons in the molecules which must be specified as a vector by its intensity per unit volume: while the movements of the electrons, free and paired, in such element of volume must be combined into statistical aggregates of translational fluxes and molecular whirls of electrification. With anything else than mean aggregates of the various types that can be thus separated out, each extended over the effective element of volume, mechanical science, which has for its object matter in bulk as it presents itself to our observation and experiment, is not directly concerned: there is however another more abstract study, that of molecular dynamics, whose province it is to form and test hypotheses of molecular structure and arrangement, intended to account for the distinctive features of the mechanical phenomena aforesaid.

As the integral $\int(lf+mg+nh)\,dS$, extended over the boundary of any region, no longer vanishes when there are electrons in that region, it follows that the vector (f, g, h) which represents the strain or "electric displacement" of the aether, is no longer circuital when these individual electrons are merged in volume-densities, as they are when we consider a material medium continuously distributed, instead of merely the aether existing between its molecules; thus the definition of the mode of change of aethereal elastic displacement namely

$$4\pi(\dot{f}, \dot{g}, \dot{h}) = \text{curl}\,(\dot{\xi}, \dot{\eta}, \dot{\zeta}),$$

which held for free aether, would now be a contradiction in terms. In order to ascertain what is to replace this definition, let us consider the translation of a single electron e from a point P_1 to a neighbouring point P_2. This will cause an addition to the elastic strain (f, g, h) of the aether, represented by a strain-vector distributed with reference to lines which begin at P_1 and end at P_2, the addition being in fact the electric displacement due to the doublet formed by $-e$ at P_1 and $+e$ at P_2. This additional flux of electric displacement from P_2 to P_1 along these lines is not by itself circuital; but the circuits of the flux will be completed if we add to it a linear flux of electricity of the same total amount e, back again from P_1 to P_2 along the line P_1P_2. If we complete in this way the fluxes of *aethereal electric displacement*, due to the changes of position of all the electrons of the system, by the fluxes of these *true electric charges* through the aether, a new vector is obtained which we may call the flux of the *total electric displacement* per unit volume; and this vector forms a fundamentally useful conception from the circumstance that it is everywhere and always a circuital or stream vector.

We may now express this result analytically: to the rate of change of aethereal displacement $(\dot{f}, \dot{g}, \dot{h})\,\delta\tau$ in the element of volume $\delta\tau$ there must be added $\Sigma(e\dot{x}, e\dot{y}, e\dot{z})$, where $(\dot{x}, \dot{y}, \dot{z})$ is the velocity of a contained electron e, in order to get a circuital result: the *current of aethereal electric displacement* by itself is not circuital when averaged with regard to this element of volume, but the so-called

total current, made up of it, and of the *true electric current* formed by the moving electrons, possesses that property.

Thus we have to deal, in the mechanical theory, with a more complex problem: instead of only aethereal displacement we have now two *independent* variables, aethereal displacement, and true electric current or flux of electrons. In the molecular analysis, on the other hand, the minute knowledge of aethereal displacement between and around the electrons of the molecules involved that of the movements of these electrons or singularities themselves, and there was only one independent variable, at any rate when the singularities are purely aethereal. The transition, from the complete knowledge of aether and individual molecules to the averaged and smoothed out specification of the element of volume of the complex medium, requires the presence of two independent variables, one for the aether and one for the matter, instead of a single variable only.

53. We may consider this fundamental explanation from a different aspect. There are present in the medium electrons or electric charges each of amount e, so that for any region Faraday's hypothesis gives

$$\int (lf+mg+nh)\,dS = \Sigma e;$$

and therefore, any finite change of state being denoted by $\triangle$, $\triangle \int (lf+mg+nh)\,dS$ is equal to the flux of electrons into the region across the boundary. Thus for example

$$\frac{d}{dt}\int (lf+mg+nh)\,dS = -\int (lu_0+mv_0+nw_0)\,dS$$

in which (u_0, v_0, w_0) is the true electric current which is simply this flux of electrons reckoned per unit time: hence transposing all the terms to the same side, we have for any closed surface

$$\int (lu+mv+nw)\, dS = 0,$$

where $\quad (u, v, w) = (df/dt+u_0,\ dg/dt+v_0,\ dh/dt+w_0).$

This relation expresses that (u, v, w), the total current of Maxwell's theory, is circuital or a stream.

The true current (u_0, v_0, w_0) above defined includes all the possible types of co-ordinated or averaged motions of electrons, namely, currents arising from conduction, from material polarization and its convection, from convection of charged bodies.

54. We have now to fix the meaning to be attached to $(\dot\xi, \dot\eta, \dot\zeta)$ or (a, b, c) in a mechanical theory which treats only of sensible elements of volume. Obviously it must be the mean value of this vector, as previously employed, for the aether in each element of volume. With this meaning it is now to be shown that the curl of $(\xi, \dot\eta, \zeta)$ is equal to $4\pi\,(u, v, w)$. We shall in fact see that for any open geometrical surface or sheet S of sensible extent, fixed in space, bounded by a contour s, Sir George Stokes' fundamental analytical theorem of transformation of a surface integral into a line integral round its contour, must under the present circumstances assume the wider form

$$\frac{1}{4\pi} \triangle \int \left(\xi \frac{dx}{ds}+\dot\eta \frac{dy}{ds}+\zeta \frac{dz}{ds}\right) ds = \triangle \int (l\dot f+m\dot g+n\dot h)\, dS+\mathfrak{F} \quad \text{(i)}$$

where the symbol $\triangle$ represents the change in the integral which follows it, produced by the motion of the system in any finite time, and $\mathfrak{F}$ represents the total flux of electrons through the fixed surface S during that time. To this end consider two sheets S and S' both abutting on the same contour s: then as the two together form a closed surface we have

$$\int (l'f+m'g+n'h)\, dS' - \int (lf+mg+nh)\, dS = \Sigma e \qquad \text{(ii)}$$

where Σe denotes the sum of the strengths of the electrons included between the sheets: in this formula the direction vectors (l', m', n') and (l, m, n) are both measured towards the same sides of the surfaces, which for the former S' is the side away from the region enclosed between them. Now if one of these included electrons moves across the surface S', the form of the integral for that surface will be abruptly altered, an element of it becoming infinite at the transition when the electron is on the surface; and this will vitiate the proof of Stokes' theorem considered as applying to the change in the value of that surface integral. But the form of the integral for the other surface, across which the electron has not penetrated, will not pass through any critical stage, and Stokes' theorem will still hold for the change caused in it. That is, for the latter surface the equation (i) will hold good in the ordinary way without any term such as $\mathfrak{F}$; and therefore by (ii), for the former surface, across which electrons are taken to pass, the term $\mathfrak{F}$ as above is involved.

The relation of Sir George Stokes, thus generalized, in which $\mathfrak{F}$ represents the total flux of electrons across the surface S, leads directly to the equation

$$\text{curl}\ (\xi, \dot\eta, \zeta) = 4\pi(\dot f+u_0,\ \dot g+v_0,\ \dot h+w_0)$$

where the vectors *now* represent mean values throughout the element of volume.

This relation holds, whether the system of molecules contained in the medium is *magnetically* polarized or not, for the transference of magnetic polarity across the sheet S cannot add anything to the electric flux through it: it appears therefore that in a case involving magnetic polarization $(\xi, \dot\eta, \zeta)$ represents what is called the magnetic induction and not the magnetic force, which is also in keeping with the stream character of the former vector. On the other hand the change in the *electric* polarization (f', g', h') of the molecules constitutes an addition $\triangle(f', g', h')$ of finite amount per unit area to the flux through the sheet, so that $d/dt\,(f', g', h')$ constitutes a part of the true electric current (u_0, v_0, w_0).

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Chapter X

GENERAL PROBLEM OF MOVING MATTER TREATED IN RELATION TO THE INDIVIDUAL MOLECULES

Formulation of the Problem

102. WE shall now consider the material system as consisting of free aether pervaded by a system of electrons which are to be treated individually, some of them free or isolated, but the great majority of them grouped into material molecules: and we shall attempt to compare the relative motions of these electrons when they form, or belong to, a material system devoid of translatory motion through the aether, with what it would be when a translatory velocity is superposed, say for shortness a velocity v parallel to the axis of x. The medium in which the activity occurs is for our present purpose the free aether itself, whose dynamical equations have been definitely ascertained in quite independent ways from consideration of both the optical side and the electrodynamic side of its activity: so that there will be nothing hypothetical in our analysis on that score. An electron e will occur in this analysis as a singular point in the aether, on approaching which the elastic strain constituting the aethereal displacement (f, g, h) increases indefinitely, according to the type

$$-e/4\pi \cdot (d/dx, \quad d/dy, \quad d/dz)r^{-1}:$$

it is in fact analogous to what is called a simple pole in the two-dimensional representation that is employed in the theory of a function of a complex variable. It is assumed that this singularity represents a definite structure, forming a nucleus of strain in the aether, which is capable of transference across that medium independently of motion of the aether itself: the portion of the surrounding aethereal strain, of which the displacement-vector (f, g, h)

is the expression, which is associated with the electron and is carried along with the electron in its motion, being as above $-e/4\pi \times(d/dx, d/dy, d/dz)r^{-1}$. It is to be noticed that the energy of this part of the displacement is closely concentrated around the nucleus of the electron, and not widely diffused as might at first sight appear. The aethereal displacement satisfies the stream-condition

$$df/dx+dg/dy+dh/dz = 0,$$

except where there are electrons in the effective element of volume: these are analogous to the so-called sources and sinks in the abstract theory of liquid flow, so that when electrons are present the integral of the normal component of the aethereal displacement over the boundary of any region, instead of being null, is equal to the quantity Σe of electrons existing in the region. The other vector which is associated with the aether, namely the magnetic induction (a, b, c), also possesses the stream property; but singular points in its distribution, of the nature of simple poles, do not exist. The motion of an electron involves however a singularity in (a, b, c), of a rotational type, with its nucleus at the moving electron;† and the time-average of this singularity for a very rapid minute steady

† Namely as the distance r from it diminishes indefinitely, the magnetic induction tends to the form $evr^{-2} \sin\theta$, at right angles to the plane of the angle θ between r and the velocity v of the electron: this arises as the disturbance of the medium involved in annulling the electron in its original position and restoring it in the new position to which it has moved. The relations will appear more clearly when visualized by the kinematic representation of Appendix E; or when we pass to the limit in the formulae of Chapter ix relating to the field of a moving charged body of finite dimensions.

The specification in the text, as a simple pole, only applies for an electron moving with velocity v, when terms of the order $(v/c)^2$ are neglected: otherwise the aethereal field close around it is not isotropic and an amended specification derivable from the formulae of Chapter ix must be substituted. In the second-order discussion of Chapter xi this more exact form is implicitly involved, the strength of the electron being determined (§ 111) by the concentration of the aethereal displacement around it. The singularity in the magnetic field which is involved in the motion of the electron, not of course an intrinsic one, has no concentration.

orbital motion of an electron is analytically equivalent, at distances considerable compared with the dimensions of the orbit, to a magnetic doublet analogous to a source and associated equal sink. Finally, the various parts of the aether are supposed to be sensibly at rest, so that for example the time-rate of change of the strain of any element of the aether is represented by differentiation with respect to the time without any additional terms to represent the change due to the element of aether being carried on in the meantime to a new position; in this respect the equations of the aether are much simpler than those of the dynamics of fluid motion, being in fact linear. The aether is stagnant on this theory, while the molecules constituting the Earth and all other material bodies flit through it without producing any finite flow in it; hence the law of the astronomical aberration of light is rigorously maintained, and the Doppler change of wave-length of radiation from a moving source holds good; but it will appear that all purely terrestrial optical phenomena are unaffected by the Earth's motion.

103. Subject to this general explanation, the analytical equations which express the dynamics of the field of free aether, existing between and around the nuclei of the electrons, are

$$4\pi \frac{d}{dt}(f, g, h) = \operatorname{curl}(a, b, c)$$

$$-\frac{d}{dt}(a, b, c) = 4\pi c^2 \operatorname{curl}(f, g, h),$$

in which the symbol curl (a, b, c) represents, after Maxwell, the vector

$$\left(\frac{dc}{dy}-\frac{db}{dz}, \frac{da}{dz}-\frac{dc}{dx}, \frac{db}{dx}-\frac{da}{dy}\right),$$

and in which c is the single physical constant of the aether, being the velocity of propagation of elastic disturbances through it. These are the analytical equations derived by Maxwell in his

mathematical development of Faraday's views as to an electric medium: and they are the same as the equations arrived at by MacCullagh a quarter of a century earlier in his formulation of the dynamics of optical media. It may fairly be claimed that the theoretical investigations of Maxwell, in combination with the experimental verifications of Hertz and his successors in that field, have imparted to this analytical formulation of the dynamical relations of free aether an exactness and precision which is not surpassed in any other department of physics, even in the theory of gravitation.

Where a more speculative element enters is in the construction of a kinematic scheme of representation of the aether-strain, such as will allow of the unification of the various assumptions here enumerated. It is desirable for the sake of further insight, and even necessary for various applications, to have concrete notions of the physical nature of the vectors (f, g, h) and (a, b, c) which specify aethereal disturbances, in the form of representations such as will implicitly and intuitively involve the analytical relations between them, and will also involve the conditions and restrictions to which each is subject, including therein the permanence and characteristic properties of an electron and its free mobility through the aether.†

104. But for the mere analytical development of the aether-scheme as above formulated, a concrete physical representation of the constitution of the aether is not required: the abstract relations and conditions above given form a sufficient basis. In point of fact these analytical relations are theoretically of an ideal simplicity for this purpose: for they give explicitly the time-rates of change of the vectors of the problem at each instant, so that from a knowledge of the state of the system at any time t the state at the time $t+\delta t$ can be immediately expressed, and so by successive steps, or by the use of Taylor's differential expansion-theorem, its state at any further time can theoretically be derived. The point that requires careful attention is as to whether the solution of these

† See Appendix E.

equations in terms of a given initial state of the system determines the motions of the electrons or strain-nuclei through the medium, as well as the changes of strain in the medium itself: and it will appear on consideration that under suitable hypotheses this is so. For the given initial state will involve given motions of the electrons, that is the initial value of (a, b, c) will involve rotational singularities at the electrons around their directions of motion, just such as in the element of time δt will shift the electrons themselves into their new positions:† and so on step by step continually. This however presupposes that the nucleus of the electron is quite labile as regards displacement through the aether, in other words that its movement is not influenced by any inertia or forces except such as are the expression of its relation to the aether: we in fact assume the *completeness* of the aethereal scheme of relations as above given. Any difficulty that may be felt on account of the infinite values of the vectors at the nucleus itself may be removed, in the manner customary in analytical discussions on attractions, by considering the nucleus to consist of a volume distribution of electricity of finite but very great density, distributed through a very small space instead of being absolutely concentrated in a point: then the quantities will not become infinite. Of the detailed structure of electrons nothing is assumed: so long as the actual dimensions of their nuclei are extremely small in comparison with the distances between them, it will suffice for the theory to consider them as points, just as for example in the general gravitational theory of the Solar System it suffices to consider the planets as attracting points. This method is incomplete only as regards those portions of the energy and other quantities that are associated with the mutual actions of the parts of the electron itself, and are thus molecularly constitutive.

105. It is to be observed that on the view here being developed, in which atoms of matter are constituted of aggregations of electrons, the only actions between atoms are what may be described as electric forces. The electric character of the forces of chemical affinity was an accepted part of the chemical views of Davy, Berzelius, and Faraday; and more recent discussions, while clearing away crude conceptions, have invariably tended to the strengthening of that hypothesis. The mode in which the ordinary forces of cohesion could be included in such a view is still quite undeveloped. Difficulties of this kind have however not been felt to be fundamental in the vortex-atom illustration of the constitution of matter, which has exercised much fascination over high authorities on molecular physics: yet in the concrete realization of Maxwell's theory of the aether above referred to, the atom of matter possesses all the dynamical properties of a vortex ring in a frictionless fluid, so that everything that can be done in the domain of vortex-ring illustration is implicitly attached to the present scheme. The fact that virtually nothing has been achieved in the department of forces of cohesion is not a valid objection to the development of a theory of the present kind. For the aim of theoretical physics is not a complete and summary conquest of the *modus operandi* of natural phenomena: that would be hopelessly unattainable if only for the reason that the mental apparatus with which we conduct the search is itself in one of its aspects a part of the scheme of Nature which it attempts to unravel. But the very fact that this is so is evidence of a correlation between the process of thought and the processes of external phenomena, and is an incitement to push on further and bring out into still clearer and more direct view their interconnexions. When we have mentally reduced to their simple elements the correlations of a large domain of physical phenomena, an objection does not lie because we do not know the way to push the same principles to the explanation of other phenomena to which they should presumably apply, but which are mainly beyond the reach of our direct examination.

The natural conclusion would rather be that a scheme, which has been successful in the simple and large-scale physical pheno-

† Cf. footnote, p. 162. [p. 225 in this version – K.F.S.]

mena that we can explore in detail, must also have its place, with proper modifications or additions on account of the difference of scale, in the more minute features of the material world as to which direct knowledge in detail is not available. And in any case, whatever view may be held as to the necessity of the whole complex of chemical reaction being explicable in detail by an efficient physical scheme, a limit is imposed when vital activity is approached: any complete analysis of the conditions of the latter, when merely superficial sequences of phenomena are excluded, must remain outside the limits of our reasoning faculties. The object of scientific explanation is in fact to coordinate mentally, but not to exhaust, the interlaced maze of natural phenomena: a theory which gives an adequate correlation of a portion of this field maintains its place until it is proved to be in definite contradiction, not removable by suitable modification, with another portion of it.

Application to moving Material Media: approximation up to first order

106. We now recall the equations of the free aether, with a view to changing from axes (x, y, z) at rest in the aether to axes (x', y', z') moving with translatory velocity v parallel to the axis of x; so as thereby to be in a position to examine how phenomena are altered when the observer and his apparatus are in uniform motion through the stationary aether. These equations are

$$4\pi\frac{df}{dt} = \frac{dc}{dy} - \frac{db}{dz} \qquad -(4\pi c^2)^{-1}\frac{da}{dt} = \frac{dh}{dy} - \frac{dg}{dz}$$

$$4\pi\frac{dg}{dt} = \frac{da}{dz} - \frac{dc}{dx} \qquad -(4\pi c^2)^{-1}\frac{db}{dt} = \frac{df}{dz} - \frac{dh}{dx}$$

$$4\pi\frac{dh}{dt} = \frac{db}{dx} - \frac{da}{dy} \qquad -(4\pi c^2)^{-1}\frac{dh}{dt} = \frac{dg}{dx} - \frac{df}{dy}.$$

When they are referred to the axes (x', y', z') in uniform motion, so that $(x', y', z') = (x-vt, y, z)$, $t' = t$, then d/dx, d/dy, d/dz become

d/dx', d/dy', d/dz', but d/dt becomes $d/dt' - vd/dx'$: thus

$$4\pi\frac{df}{dt'} = \frac{dc'}{dy'} - \frac{db'}{dz'} \qquad -(4\pi c^2)^{-1}\frac{da}{dt'} = \frac{dh'}{dy'} - \frac{dg'}{dz'}$$

$$4\pi\frac{dg}{dt'} = \frac{da'}{dz'} - \frac{dc'}{dx'} \qquad -(4\pi c^2)^{-1}\frac{db}{dt'} = \frac{df'}{dz'} - \frac{dh'}{dx'}$$

$$4\pi\frac{dh}{dt'} = \frac{db'}{dx'} - \frac{da'}{dy'} \qquad -(4\pi c^2)^{-1}\frac{dc}{dt'} = \frac{dg'}{dx'} - \frac{df'}{dy'},$$

where

$$(a', b', c') = (a, b+4\pi vh, c-4\pi vg)$$

$$(f', g', h') = \left(f, g-\frac{v}{4\pi c^2}c, h+\frac{v}{4\pi c^2}b\right).$$

We can complete the elimination of (f, g, h) and (a, b, c) so that only the vectors denoted by accented symbols shall remain, by substituting from these latter formulae: thus

$$g = g' + \frac{v}{4\pi c^2}(c'+4\pi vg),$$

so that $$\varepsilon^{-1}g = g' + \frac{v}{4\pi c^2}c',$$

where ε is equal to $(1-v^2/c^2)^{-1}$, and exceeds unity;

and $$b = b' - 4\pi v\left(h' - \frac{v}{4\pi c^2}b\right)$$

so that $$\varepsilon^{-1}b = b' - 4\pi vh';$$

giving the general relations

$$\varepsilon^{-1}(a, b, c) = (\varepsilon^{-1}a', b'-4\pi vh', c'+4\pi vg')$$

$$\varepsilon^{-1}(f, g, h) = \left(\varepsilon^{-1}f', g'+\frac{v}{4\pi c^2}c', h'-\frac{v}{4\pi c^2}b'\right).$$

Hence

$$4\pi \frac{df'}{dt'} = \frac{dc'}{dy'} - \frac{db'}{dz'}$$

$$4\pi\varepsilon \frac{dg'}{dt'} = \frac{da'}{dz'} - \left(\frac{d}{dx'} + \frac{v}{c^2}\varepsilon\frac{d}{dt'}\right)c'$$

$$4\pi\varepsilon \frac{dh'}{dt'} = \left(\frac{d}{dx'} + \frac{v}{c^2}\varepsilon\frac{d}{dt'}\right)b' - \frac{da'}{dy'}$$

$$-(4\pi c^2)^{-1} \frac{da'}{dt'} = \frac{dh'}{dy'} - \frac{dg'}{dz'}$$

$$-(4\pi c^2)^{-1}\varepsilon \frac{db'}{dt'} = \frac{df'}{dz'} - \left(\frac{d}{dx'} + \frac{v}{c^2}\varepsilon\frac{d}{dt'}\right)h'$$

$$-(4\pi c^2)^{-1}\varepsilon \frac{dc'}{dt'} = \left(\frac{d}{dx'} + \frac{v}{c^2}\varepsilon\frac{d}{dt'}\right)g' - \frac{df'}{dy'}.$$

Now change the time-variable from t' to t'', equal to $t'-(v/c^2)\varepsilon x'$; this will involve that

$$\frac{d}{dx'} + \frac{v}{c^2}\varepsilon\frac{d}{dt'}$$

is replaced by d/dx', while the other differential operators remain unmodified; thus the scheme of equations reverts to the same type as when it was referred to axes at rest, except as regards the factors ε on the left-hand sides.

107. It is to be observed that this factor ε only differs from unity by $(v/c)^2$, which is of the second order of small quantities; hence we have the following correspondence when that order is neglected. Consider any aethereal system, and let the sequence of its spontaneous changes referred to axes (x', y', z') moving uniformly through the aether with velocity $(v, 0, 0)$ be represented by values of the vectors (f, g, h) and (a, b, c) expressed as functions of x', y', z' and t', the latter being the time measured in the ordinary manner: then there exists a correlated aethereal system whose sequence of spontaneous changes referred to axes (x', y', z') at rest are such that its electric and magnetic vectors (f', g', h') and (a', b', c') are functions of the variables x', y', z' and a time-variable t'', equal to $t'-(v/c^2)x'$, which are the same as represent the quantities

$$\left(f,\ g-\frac{v}{4\pi c^2}c,\ h+\frac{v}{4\pi c^2}b\right)$$

and $$(a,\ b+4\pi vh,\ c-4\pi vg)$$

belonging to the related moving system when expressed as functions of the variables x', y', z' and t'.

Conversely, taking any aethereal system at rest in the aether, let the sequence of its changes be represented by (f', g', h') and (a', b', c') expressed as functions of the co-ordinates (x, y, z) and of the time t'. In these functions change t' into $t-(v/c^2)x$: then the resulting expressions are the values of

$$\left(f,\ g-\frac{v}{4\pi c^2}c,\ h+\frac{v}{4\pi c^2}b\right),$$

and $$(a,\ b+4\pi vh,\ c-4\pi vg),$$

for a system in uniform motion through the aether, referred to axes (x, y, z) moving along with it, and to the time t. In comparing the states of the two systems, we have to the first order

$$\frac{df}{dx} \quad \text{equal to} \quad \frac{df'}{dx} - \frac{v}{c^2}\frac{df'}{dt}$$

$$\frac{d}{dy}\left(g-\frac{v}{4\pi c^2}c\right) \quad \text{equal to} \quad \frac{dg'}{dy}$$

$$\frac{d}{dz}\left(h+\frac{v}{4\pi c^2}b\right) \quad \text{equal to} \quad \frac{dh'}{dz};$$

hence bearing in mind that for the system at rest

$$\frac{dc'}{dy} - \frac{db'}{dz} = 4\pi\frac{df'}{dt'},$$

or, what is the same,

$$\frac{dc'}{dy}-\frac{db'}{dz}=4\pi\left(\frac{df'}{dt}-v\frac{df'}{dx}\right),$$

we have, to the first order,

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=\frac{df'}{dx}+\frac{dg'}{dy}+\frac{dh'}{dz}.$$

Thus the electrons in the two systems here compared, being situated at the singular points at which the concentration of the electric displacement ceases to vanish, occupy corresponding positions. Again, these electrons are of equal strengths: for, very near an electron, fixed or moving, the values of (f, g, h) and (a, b, c) are practically those due to it, the part due to the remainder of the system being negligible in comparison: also in this correspondence the relation between (f, g, h) and the accented variables is, by § 106

$$\varepsilon^{-1}(f, g, h)=\left(\varepsilon^{-1}f', g'+\frac{v}{4\pi c^2}c', h'-\frac{v}{4\pi c^2}b'\right);$$

hence, since for the single electron at rest (a', b', c') is null, we have, very close to the correlative electron in the moving system, (f, g, h) equal to $(f', \varepsilon g', \varepsilon h')$, where ε, being $(1-v^2/c^2)^{-1}$, differs from unity by the second order of small quantities. Thus neglecting the second order, (f, g, h) is equal to (f', g', h') for corresponding points very close to electrons; and, as the amount of electricity inside any boundary is equal to the integral of the normal component of the aethereal displacement taken over the boundary, it follows by taking a very contracted boundary that the strengths of the corresponding electrons in the two systems are the same, to this order of approximation.

108. It is to be observed that the above analytical transformation of the equations applies to any isotropic dielectric medium as well as to free aether: we have only to alter c into the velocity of radiation in that medium, and all will be as above. The transformation will thus be different for different media. But we are arrested if we attempt to proceed to compare a moving material system, treated as continuous, with the same system at rest; for the motion of the polarized dielectric matter has altered the mathematical type of the electric current. It is thus of no avail to try to effect in this way a direct general transformation of equations of a material medium in which dielectric and conductive coefficients occur.

109. The correspondence here established between a system referred to fixed axes and a system referred to moving axes will assume a very simple aspect when the former system is a steady one, so that the variables are independent of the time. Then the distribution of electrons in the second system will be at each instant precisely the same as that in the first, while the second system accompanies its axes of reference in their uniform motion through the aether. In other words, given any system of electrified bodies at rest, in equilibrium under their mutual electric influences and imposed constraints, there will be a precisely identical system in equilibrium under the same constraints, and in uniform translatory motion through the aether. That is, uniform translatory motion through the aether does not produce any alteration in electric distributions as far as the first order of the ratio of the velocity of the system to the velocity of radiation is concerned. Various cases of this general proposition will be verified subsequently in connexion with special investigations.

Moreover this result is independent of any theory as to the nature of the forces between material molecules: the structure of the matter being assumed unaltered to the first order by motion through the aether, so too must be all electric distributions. What has been proved comes to this, that if any configuration of ionic charges is the natural one in a material system at rest, the maintenance of the same configuration as regards the system in uniform motion will not require the aid of any new forces. The electron *taken by itself* must be on any conceivable theory a simple singularity of the aether whose movements when it is free, and interactions with other electrons if it can be constrained by matter, are traceable through the differential equations of the surrounding free aether alone: and a correlation has been established between these equations for the two cases above compared. It is however to be observed (cf. § 99) that though the fixed and the moving system of electrons of this correlation are at corresponding instants identical, yet the electric and magnetic displacements belonging to them differ by terms of the first order.

Chapter XI

MOVING MATERIAL SYSTEM: APPROXIMATION CARRIED TO THE SECOND ORDER

110. The results above obtained have been derived from the correlation developed in § 106, up to the first order of the small quantity v/c, between the equations for aethereal vectors here represented by (f', g', h') and (a', b', c') referred to the axes (x', y', z') at rest in the aether and a time t'', and those for related aethereal vectors represented by (f, g, h) and (a, b, c) referred to axes (x', y', z') in uniform translatory motion and a time t'. But we can proceed further, and by aid of a more complete transformation institute a correspondence which will be correct to the second order. Writing as before t'' for $t'-(v/c^2)\varepsilon x'$, the exact equations for (f, g, h) and (a, b, c) referred to the moving axes (x', y', z') and time t' are, as above shown, equivalent to

$$4\pi\,\frac{df'}{dt''} = \frac{dc'}{dy'} - \frac{db'}{dz'} \qquad -(4\pi c^2)^{-1}\,\frac{da'}{dt''} = \frac{dh'}{dy'} - \frac{dg'}{dz'}$$

$$4\pi\varepsilon\,\frac{dg'}{dt''} = \frac{da'}{dz'} - \frac{dc'}{dx'} \qquad -(4\pi c^2)^{-1}\varepsilon\,\frac{db'}{dt''} = \frac{df'}{dz'} - \frac{dh'}{dx'}$$

$$4\pi\varepsilon\,\frac{dh'}{dt''} = \frac{db'}{dx'} - \frac{da'}{dy'} \qquad -(4\pi c^2)^{-1}\varepsilon\,\frac{dc'}{dt''} = \frac{dg'}{dx'} - \frac{df'}{dy'}\,.$$

Now write

$$(x_1, y_1, z_1) \quad \text{for} \quad \left(\varepsilon^{\frac{1}{2}}x', y', z'\right)$$

$$(a_1, b_1, c_1) \quad \text{for} \quad \left(\varepsilon^{-\frac{1}{2}}a', b', c'\right) \quad \text{or} \quad \left(\varepsilon^{-\frac{1}{2}}a, +4\pi vh, c-4\pi vg\right)$$

$$(f_1, g_1, h_1) \quad \text{for} \quad \left(\varepsilon^{-\frac{1}{2}}f', g', h'\right)$$

or

$$\left(\varepsilon^{-\frac{1}{2}}f, g-\frac{v}{4\pi c^2}\,c,\ h+\frac{v}{4\pi c^2}\,b\right)$$

$$dt_1 \quad \text{for} \quad \varepsilon^{-\frac{1}{2}}\,dt'' \qquad \text{or} \quad \varepsilon^{-\frac{1}{2}}\left(dt'-\frac{v}{c^2}\,\varepsilon dx'\right),$$

where $\varepsilon = (1-v^2/c^2)^{-1}$; and it will be seen that the factor ε is absorbed, so that the scheme of equations, referred to moving axes, which connects together the new variables with subscripts, is identical in form with the Maxwellian scheme of relations for the aethereal vectors referred to fixed axes. This transformation, from (x', y', z') to (x_1, y_1, z_1) as dependent variables, signifies an elongation of the space of the problem in the ratio $\varepsilon^{\frac{1}{2}}$ along the direction of the motion of the axes of coordinates. Thus if the values of (f_1, g_1, h_1) and (a_1, b_1, c_1) given as functions of x_1, y_1, z_1, t_1 express the course of spontaneous change of the aethereal vectors of a system of moving electrons referred to axes (x_1, y_1, z_1) at rest in the aether, then

$$\left(\varepsilon^{-\frac{1}{2}}f, g-\frac{v}{4\pi c^2}\,c,\ h+\frac{v}{4\pi c^2}\,b\right)$$

and

$$\left(\varepsilon^{-\frac{1}{2}}a,\ b+4\pi vh,\ c-4\pi vg\right),$$

expressed by the same functions of the variables

$$\varepsilon^{\frac{1}{2}}x', y', z', \varepsilon^{-\frac{1}{2}}t'-\frac{v}{c^2}\,\varepsilon^{\frac{1}{2}}x',$$

will represent the course of change of the aethereal vectors (f, g, h) and (a, b, c) of a correlated system of moving electrons referred to axes of (x', y', z') moving through the aether with uniform translatory velocity $(v, 0, 0)$. In this correlation between the courses of change of the two systems, we have

$$\frac{d\left(\varepsilon^{-\frac{1}{2}}f\right)}{d\left(\varepsilon^{\frac{1}{2}}x'\right)} \quad \text{equal to} \quad \frac{df_1}{dx_1}-\frac{v}{c^2}\frac{df_1}{dt_1},$$

$$\frac{d}{dy'}\left(g-\frac{v}{4\pi c^2}c\right) \quad \text{equal to} \quad \frac{dg_1}{dy_1}$$

$$\frac{d}{dz'}\left(h+\frac{v}{4\pi c^2}b\right) \quad \text{equal to} \quad \frac{dh_1}{dz_1},$$

where
$$\frac{dc}{dy'}-\frac{db}{dz'} = 4\pi\left(\frac{df}{dt'}-v\frac{df}{dx'}\right)$$

and also
$$\frac{df_1}{dt_1} = \frac{df}{dt};$$

hence
$$\frac{df}{dx'}+\frac{dg}{dy'}+\frac{dh}{dz'}-\frac{v}{c^2}\left(\frac{df}{dt'}-v\frac{df}{dx'}\right)$$

is equal to
$$\varepsilon\frac{df_1}{dx_1}+\frac{dg_1}{dy_1}+\frac{dh_1}{dz_1}-\frac{v}{c^2}\varepsilon\frac{df}{dt},$$

so that, up to the order of $(v/c)^2$ inclusive,

$$\frac{df}{dx'}+\frac{dg}{dy'}+\frac{dh}{dz'} = \frac{df_1}{dx_1}+\frac{dg_1}{dy_1}+\frac{dh_1}{dz_1}.$$

Thus the conclusions as to the corresponding positions of the electrons of the two systems, which had been previously established up to the first order of v/c, are true up to the second order when the dimensions of the moving system are contracted in comparison with the fixed system in the ratio $\varepsilon^{-\frac{1}{2}}$, or $1-\frac{1}{2}v^2/c^2$, along the direction of its motion.

111. The ratio of the strengths of corresponding electrons in the two systems may now be deduced just as it was previously when the discussion was confined to the first order of v/c. For the case of a single electron in uniform motion the comparison is with a single electron at rest, near which (a_1, b_1, c_1) vanishes so far as it depends on that electron: now we have in the general correlation

$$g = g_1+\frac{v}{4\pi c^2}(c_1+4\pi v g),$$

hence in this particular case

$$(g, h) = \varepsilon(g_1, h_1), \quad \text{while} \quad f = \varepsilon^{\frac{1}{2}}f_1.$$

But the strength of the electron in the moving system is the value of the integral

$$\iint (f\,dy'\,dz'+g\,dz'\,dx'+h\,dx'\,dy')$$

extended over any surface closely surrounding its nucleus; that is here

$$\varepsilon^{\frac{1}{2}}\iint (f_1\,dy_1\,dz_1+g_1\,dz_1\,dx_1+h_1\,dx_1\,dy_1),$$

so that the strength of each moving electron is $\varepsilon^{\frac{1}{2}}$ times that of the correlative fixed electron. As before, no matter what other electrons are present, this argument still applies if the surface be taken to surround the electron under consideration very closely, because then the wholly preponderating part of each vector is that which belongs to the adjacent electron.†

112. We require however to construct a correlative system devoid of the translatory motion in which the strengths of the elect-

† This result follows more immediately from § 110, which shows that corresponding densities of electrification are equal, while corresponding volumes are as $\varepsilon^{\frac{1}{2}}$ to unity.

rons shall be equal instead of proportional, since motion of a material system containing electrons cannot alter their strengths. The principle of dynamical similarity will effect this.

We have in fact to reduce the scale of the electric charges, and therefore of

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz},$$

in a system at rest in the ratio $\varepsilon^{-\frac{1}{2}}$. Apply therefore a transformation

$$(x,\ y,\ z) = k(x_1,\ y_1,\ z_1), \qquad t = lt_1,$$

$$(a,\ b,\ c) = \vartheta(a_1,\ b_1,\ c_1),\ (f,\ g,\ h) = \varepsilon^{-\frac{1}{2}}k(f_1,\ g_1,\ h_1);$$

and the form of the fundamental circuital aethereal relations will not be changed provided $k = l$ and $\vartheta = \varepsilon^{-\frac{1}{2}}k$. Thus we may have k and l both unity and $\vartheta = \varepsilon^{-\frac{1}{2}}$; so that no further change of scale in space and time is required, but only a diminution of (a, b, c) in the ratio $\varepsilon^{-\frac{1}{2}}$.

We derive the result, correct to the second order, that if the internal forces of a material system arise wholly from electrodynamic actions between the systems of electrons which constitute the atoms, then an effect of imparting to a steady material system a uniform velocity of translation is to produce a uniform contraction of the system in the direction of the motion, of amount $\varepsilon^{-\frac{1}{2}}$ or $1-\frac{1}{2}v^2/c^2$. The electrons will occupy corresponding positions in this contracted system, but the aethereal displacements in the space around them will not correspond: if (f, g, h) and (a, b, c) are those of the moving system, then the electric and magnetic displacements at corresponding points of the fixed systems will be the values that the vectors

$$\varepsilon^{\frac{1}{2}}\left(\varepsilon^{-\frac{1}{2}}f,\ g-\frac{v}{4\pi c^2}c,\ h+\frac{v}{4\pi c^2}b\right)$$

and
$$\varepsilon^{\frac{1}{2}}\left(\varepsilon^{-\frac{1}{2}}a,\ b+4\pi vh,\ c-4\pi vg\right)$$

had at a time const.$+vx/c^2$ before the instant considered when the scale of time is enlarged in the ratio $\varepsilon^{\frac{1}{2}}$.

As both the electric and magnetic vectors of radiation lie in the wave-front, it follows that in the two correlated systems, fixed and moving, the relative wave-fronts of radiation correspond, as also do the rays which are the paths of the radiant energy relative to the systems. The change of the time variable, in the comparison of radiations in the fixed and moving systems, involves the Doppler effect on the wave-length.

The Correlation between a stationary and a moving Medium, as regards trains of Radiation

113. Consider the aethereal displacement given by

$$(f_1,\ g_1,\ h_1) = (L,\ M,\ N)\,F(lx_1+my_1+nz_1-pt),$$

which belongs to a plane wave-train advancing, along the direction (l, m, n) with velocity V, or c/μ where μ is refractive index, equal to

$$p(l^2+m^2+n^2)^{-\frac{1}{2}},$$

in the material medium at rest referred to coordinates (x_1, y_1, z_1). In the corresponding wave-train relative to the same medium in motion specified by coordinates (x, y, z), and considered as shrunk in the above manner as a result of the motion, the vectors (f, g, h) and (a, b, c) satisfy the relation

$$\varepsilon^{\frac{1}{2}}\left(\varepsilon^{-\frac{1}{2}}f,\ g-\frac{v}{4\pi c^2}c,\ h+\frac{v}{4\pi c^2}b\right)$$

$$= (L,\ M,\ N)\,F\left\{l\varepsilon^{\frac{1}{2}}x+my+nz-p\varepsilon^{-\frac{1}{2}}\left(t-\frac{v}{c^2}\,\varepsilon x\right)\right\}$$

$$= (L,\ M,\ N)\,F\left\{\left(l\varepsilon^{\frac{1}{2}}+\frac{pv}{c^2}\,\varepsilon^{\frac{1}{2}}\right)x+my+nz-p\varepsilon^{-\frac{1}{2}}t\right\}.$$

As the wave-train in the medium at rest is one of transverse displacement, so that the vectors (f_1, g_1, h_1) and (a_1, b_1, c_1) are both in the wave-front, the same is therefore true for the vectors (f, g, h) and (a, b, c) in the correlative wave-train in the moving system, as was in fact to be anticipated from the circuital quality of these vectors: the direction vector of the front of the latter train is proportional to

$$\left(l\varepsilon^{\frac{1}{2}}+\frac{pv}{c^2}\,\varepsilon^{\frac{1}{2}},\ m,\ n\right),$$

and its velocity of propagation is

$$p\varepsilon^{-\frac{1}{2}}\Big/\left\{\left(l\varepsilon^{\frac{1}{2}}+\frac{pv}{c^2}\,\varepsilon^{\frac{1}{2}}\right)^2+m^2+n^2\right\}^{\frac{1}{2}}.$$

Thus, when the wave-train is travelling with velocity V along the direction of translation of the material medium, that is along the axis of x so that m and n are null, the velocity of the train relative to the moving medium is

$$V\varepsilon^{-1}\Big/\left(1+\frac{Vv}{c^2}\right),$$

which is, to the second order,

$$V\left(1-\frac{v^2}{c^2}\right)\Big/\left(1+\frac{Vv}{c^2}\right) \quad \text{or} \quad V-\frac{v}{\mu^2}-\left(\frac{1}{\mu}-\frac{1}{\mu^3}\right)\frac{v^2}{c}.$$

The second term in this expression is the Fresnel effect, and the remaining term is its second order correction on our hypothesis which includes Michelson's negative result.

In the general correlation, the wave-length in the train of radiation relative to the moving material system differs from that in the corresponding train in the same system at rest by the factor

$$\left(1+2l\,\frac{pv}{c^2}\right)^{-\frac{1}{2}}, \quad \text{or} \quad 1-lv/\mu c,$$

where l is the cosine of the inclination of the ray to the direction of v; it is thus shorter by a quantity of the first order, which represents the Doppler effect on wave-length because the period is the same up to that order.

When the wave-fronts relative to the moving medium are travelling in a direction making an angle θ', in the plane xy so that n is null, with the direction of motion of the medium, the velocity V' of the wave-train (of wave-length thus altered) relative to the medium is given by

$$\frac{\cos\theta'}{V'}=\frac{l\varepsilon}{p}+\frac{v\varepsilon}{c^2}, \qquad \frac{\sin\theta'}{V'}=\frac{m\varepsilon^{\frac{1}{2}}}{p},$$

where $(l^2+m^2)/p^2 = V^{-2}$. Thus

$$\left(\frac{\varepsilon^{-1}\cos\theta'}{V'}-\frac{v}{c^2}\right)^2+\frac{\varepsilon^{-1}\sin^2\theta'}{V'^2}=\frac{1}{V^2},$$

so that neglecting $(v/c)^3$,

$$V' = V-\frac{v}{\mu^2}\cos\theta'-\frac{1}{2}\,(1-\mu^{-2})\,\frac{v^2}{\mu c}\,(1+3\cos^2\theta'),$$

where $\mu = c/V$, of which the last term is the general form of the second order correction to Fresnel's expression. In free aether, for which μ is unity, this formula represents the velocity relative to the moving axes of an unaltered wave-train, as it ought to do.

As (f, g, h) and (a, b, c) are in the same phase in the free transparent aether, when one of them is null so is the other: hence in any experimental arrangement, regions where there is no disturbance in the one system correspond to regions where there is no disturbance in the other. As optical measurements are usually made by the null method of adjusting the apparatus so that the disturbance vanishes, this result carries the general absence of effect of the Earth's motion in optical experiments, up to the second order of small quantities.

Influence of translatory motion on the Structure of a Molecule: the law of Conservation of Mass

114. As a simple illustration of the general molecular theory, let us consider the group formed of a pair of electrons of opposite signs describing steady circular orbits round each other in a position of rest:† we can assert from the correlation, that when this pair is moving through the aether with velocity v in a direction lying in the plane of their orbits, these orbits relative to the translatory motion will be flattened along the direction of v to ellipticity $1-\frac{1}{2}v^2/c^2$, while there will be a first-order retardation of phase in each orbital motion when the electron is in front of the mean position combined with acceleration when behind it so that on the whole the period will be changed only in the second-order ratio $1+\frac{1}{2}v^2/c^2$. The specification of the orbital modification produced by the translatory motion, for the general case when the direction of that motion is inclined to the plane of the orbit, may be made similarly: it can also be extended to an ideal molecule constituted of any orbital system of electrons however complex. But this statement implies that the nucleus of the electron is merely a singular point in the aether, that there is nothing involved in it of the nature of inertia foreign to the aether: it also implies that there are no forces between the electrons other than those that exist through the mediation of the aether as here defined, that is other than electric forces.

The circumstance that the changes of their free periods, arising from convection of the molecules through the aether, are of the second order in v/c, is of course vital for the theory of the spectroscopic measurement of celestial velocities in the line of sight. That conclusion would however still hold good if we imagined the molecule to have inertia and potential energy extraneous to (*i.e.* unconnected with) the aether of optical and electrical phenomena,

† The orbital velocities are in this illustration supposed so small that radiation is not important. Cf. §§ 151–6 *infra*.

provided these properties are not affected by the uniform motion: for the aethereal fields of the moving electric charges, free or constrained, existing in the molecule, will be symmetrical fore and aft and unaltered to the first order by the motion, and therefore a change of sign of the velocity of translation will not affect them, so that the periods of free vibration cannot involve the first power of this velocity.

115. The fact that uniform motion of the molecule through the aether does not disturb its constitution to the first order, nor the aethereal symmetry of the moving system fore and aft, shows that when steady motion is established the mean kinetic energy of the system consists of the internal energy of the molecule, which is the same as when it is at rest, together with the sum of the energies belonging to the motions of translation of its separate electrons. This is verified on reflecting that the disturbance in the aether is made up additively of those due to the internal motions of the electrons in the molecule and those due to their common velocity of translation. Thus in estimating the mean value of the volume-integral of the square of the aethereal disturbance, which is the total kinetic energy, we shall have the integrated square of each of these disturbances separately, together with the integral of terms involving their product. Now one factor of this product is constant in time and symmetrical fore and aft as regards each electron, that factor namely which arises from the uniform translation; the other factor, arising from the orbital motions of the electrons, is oscillatory and symmetrical in front and rear of each orbit: thus the integrated product is by symmetry null. This establishes the result stated, that the kinetic energy of the moving molecule is made up of an internal energy, the same up to the first order of the ratio of its velocity to that of radiation as if it were at rest, and the energy of translation of its electrons. The coefficient of half the square of the velocity of translation in the latter part is therefore, up to that order, the measure of the inertia, or mass, of the molecule thus constituted. Hence when the square of the ratio

of the velocity of translation of the molecule to that of radiation is neglected, its electric inertia is equal to the sum of those of the electrons which compose it; and the fundamental chemical law of the constancy of mass throughout molecular transformations is verified for that part of the mass (whether it be all of it or not) that is of electric origin.

116. Objection has been taken to the view that the whole of the inertia of a molecule is associated with electric action, on the ground that gravitation, which has presumably no relations with such action, is proportional to mass: it has been suggested that inertia and gravity may be different results of the same cause. Now the inertia is by definition the coefficient of half the square of the velocity in the expression for the translatory energy of the molecule: in the constitution of the molecule it is admitted, from electrolytic considerations, that electric forces or agencies prevail enormously over gravitative ones: it seems fair to conclude that of its energy the electric part prevails equally over the gravitative part: but this is simply asserting that inertia is mainly of electric, or rather of aethereal, origin. Moreover the increase of kinetic electric energy of an electron arising from its motion with velocity v depends on v^2/c^2, on the coefficient of inertia of the aether, and on the dimensions of its nucleus, where c is the velocity of radiation: the increase of its gravitational energy would presumably in like manner depend on v^2/c'^2, where c' is the velocity of propagation of gravitation and is enormously greater than c. On neither ground does it appear likely that mass is to any considerable degree an attribute of gravitation.

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10. AN INQUIRY INTO ELECTRICAL AND OPTICAL PHENOMENA IN MOVING BODIES*

H. A. LORENTZ

Introduction

§ 1. No answer satisfactory to all physicists has yet been found to the question whether or not the ether takes part in the movement of ponderable bodies.

To make a decision one must in the first instance rely on the aberration of light and the phenomena connected with it. However, so far neither of the two competing theories, either that of Fresnel or that of Stokes, have been completely successful with respect to all observations. Thus the choice between the two views must be made by weighing the remaining difficulties of one against those of the other. This is the way in which I came to the opinion quite some time ago that Fresnel's idea, hypothesizing a motionless ether, is on the right path. Of course, hardly more than one objection can be raised against the theory of Mr. Stokes, namely that his assumptions about the movement of the ether taking place in the neighbourhood of the earth contradict themselves.† How-

* Published by Brill, Leyden, 1895.

† Lorentz, De l'influence du mouvement de la terre sur les phénomènes lumineux. *Arch. néerl.* **21**, 103 (1887); Lodge, Aberration problems. *London Phil. Trans.* (A), **184**, 727 (1893); Lorentz, De aberratietheorie van Stokes. *Zittingsverslagen der Akad. v. Wet. te Amsterdam*, 97 (1892–3).

ever, that is a very weighty objection and I really cannot see how one can eliminate it.

Difficulties arise for the Fresnel theory in connection with the well known interference experiment of Mr. Michelson† and according to the opinion of some, also through the experiments with which Mr. Des Coudres has tried in vain to prove the influence of the movement of the earth on the mutual induction of two circuits.‡ The results of the American scientist, however, can be explained with the aid of an auxiliary hypothesis and those found by Mr. Des Coudres can even be explained quite easily without such an auxiliary hypothesis.

As regards the observations of Mr. Fizeau§ on the rotation of the plane of polarization in glass columns, that is another matter. At first glance, the result goes definitely against Stokes' theory. But when I attempted to develop Fresnel's theory further I also encountered difficulties in explaining Fizeau's experiments, and so I came gradually to suspect that the result of the experiments arose out of observational errors, or at least that the result did not correspond with the theoretical considerations on which the experiments were based. Mr. Fizeau has been kind enough to notify my colleague Mr. van de Sande-Bakhuizen, on the latter's inquiries, that at present he himself does not consider his observations to be decisive anymore.

In the course of this work I will return to the here mentioned questions in more detail. At the moment I am only interested in justifying for the time being the point of view which I have taken.

Several well known reasons can be cited in support of the Fresnel theory. First of all there is the impossibility of confining the ether between solid or liquid walls. As far as we know, a space contain-

† Michelson, *American Journal of Science* (3), **22**, 120; **34**, 333 (1887); *Phil. Mag.* (5), **24**, 449 (1887).

‡ Des Coudres, *Wied. Ann.* **38**, 71 (1889).

§ Fizeau, *Ann. de chim. et de phys.* (3), **58**, 129 (1860); *Pogg. Ann.* **114**, 554 (1861).

ing no air behaves, mechanically speaking, like a true vacuum where the movement of ponderable bodies is concerned. Seeing how the mercury in a barometer rises to the top of the tube when it is inclined or how easily a thin walled closed metal tube can be compressed, one cannot avoid the inference that solid and liquid bodies are completely permeable to the ether. One can hardly assume that this medium could undergo compression without offering resistance.

Fizeau's famous interference experiment with flowing water† proves that transparent substances can move without imparting their full velocity to the ether contained in them. It would have been impossible for this experiment—later repeated by Michelson and Morley on a larger scale‡—to have the observed result if *everything* present in one of the tubes were to have the same velocity. Only the behaviour of opaque substances and very extensive bodies remains problematic after this.

It is to be noted furthermore that the permeability of a body for the ether can be conceived in two ways. Firstly this property may be lacking in the single atom and yet appear in a larger mass if the atoms are extremely small compared with the spaces in between them. Secondly one can assume (and this is the hypothesis which I will use as a basis for my considerations) that ponderable matter is *absolutely* permeable, i.e., that the atom and the ether exist in the same place, which is conceivable if one could regard the atoms as local modifications of the ether.

It is not my intention to enter into such speculations in more detail or to make conjectures about the nature of the ether. My wish is only to keep myself as free as possible from pre-conceived opinions on this medium or to attribute to it, for example, any of the properties of ordinary liquids and gases. Should it turn out

† Fizeau, *Ann. de chim. et de phys.* (3), **57**, 385 (1859); *Pogg. Ann.* **3**, 457 (1853).

‡ Michelson and Morley, *American Journal of Science* (3), **31**, 377 (1886).

that a description of the phenomena were to succeed best by assuming absolute permeability, then one ought to accept such an assumption for the present and leave it to future research to lead possibly to more profound understanding.

It is surely self-evident that there can be no question concerning the *absolute* rest of the ether. This expression would not even make sense. If I say for brevity's sake that the ether is at rest I only mean to say that one part of this medium does not move with respect to the other and that all observable motion of celestial bodies is motion relative to the ether.

§ 2. Since Maxwell's views have been increasingly accepted, the question about the behaviour of the ether has also become of great importance for the electromagnetic theory. In fact, strictly speaking, no experiment involving the motion of a charged body or an electrical conductor can be treated rigorously unless at the same time a statement is made about rest or motion of the ether. For an electric phenomenon the question arises whether an influence through the movement of the earth is to be expected, and, as regards the effect of the latter on optical phenomena, it is to be required of the electromagnetic theory of light that it takes into account all facts so far ascertained.

This is because the theory of aberration does not belong to those parts of Optics for the treatment of which the general principles of the wave theory are sufficient. As soon as a telescope is used one cannot get around applying Fresnel's convection coefficient to the lenses, and the value of this coefficient can only be derived from special assumptions concerning the nature of light waves.

The electromagnetic theory of light does in fact lead to the coefficient assumed by Fresnel as I demonstrated two years ago.[†] Since then I have simplified the theory considerably as well as extending it to phenomena of reflexion and refraction and also to

† Lorentz, *La théorie électromagnétique de Maxwell et son application aux corps mouvants*. Leiden, E. J. Brill, 1892.

double refracting substances.[†] I might therefore be allowed to return again to the subject.

In order to arrive at the fundamental equations for the electric phenomena in moving bodies I have taken a view which has been adopted by several physicists in the last years, i.e., I have assumed that all substances contain small electrically charged mass-particles and that all electric phenomena are based on the structure and movement of the "ions". This concept is, with respect to the electrolytes, generally recognized as the only possible one and Messrs. Giese,[‡] Schuster,[§] Arrhenius,[||] Elster and Geitel[††] have defended the view that also where the electric conductivity in gases is concerned one is dealing with a convection of ions. I do not see any obstacle to the assumption that the molecules of ponderable di-electric substances also contain such particles which are bound to definite equilibrium positions, and which can only be displaced from them through external electrical forces. This then would constitute the di-electric polarization of such substances.

The periodically alternating polarizations which according to Maxwell's theory constitute a ray of light, are attributed on this view to vibrations of the ions. It is well known that many scientists who accepted the view of the older theory of light considered the resonance of ponderable matter as the cause of dispersion, and this explanation can, in the main, be assimilated into the electromagnetic theory of light. In order to do this it is only necessary to ascribe a definite mass to the ions. I have shown this in a former treatise[‡‡] in which, it is true, I derived the fluctuation equation of

† Preliminary communications about this have appeared in the *Zittingsverlangen der Akad. v. Wet. te Amsterdam*, **28**, 149 (1892–3).

‡ Giese, *Wied. Ann.* **17**, 538 (1882).

§ Schuster, *Proc. Roy. Soc.* **37**, 317 (1884).

|| Arrhenius, *Wied. Ann.* **32**, 565 (1887); **33**, 638 (1888).

†† Elster and Geitel, *Wiener Sitz-Ber.* **97**, (2), 1255 (1888).

‡‡ Lorentz, *Over het verband tusschen de voortplantingssnelheid van het licht on de dichtheid en samenstelling der middenstoffen. Verhandelingen der Akad. van Wet. te Amsterdam*, Deel 18, 1878; *Wied. Ann.* **9**, 641 (1880).

motion from action at a distance and not from Maxwellian concepts, which I now consider much simpler. Von Helmholtz† subsequently based his electromagnetic theory of dispersion on the same idea.‡

Mr. Giese§ has applied to several cases the hypothesis that in metal conductors, too, the electricity is bound to ions. However, the picture which he draws of the phenomena in these substances differs in one point essentially from the concept which one has of conduction in electrolytes. While the particles of a dissolved salt—however frequently they may be arrested by the water molecules—can, after all, wander over long distances, the ions in the copper wire can hardly possess such great mobility. One can, however, be satisfied with a back and forth movement over molecular distances, provided one assumes that frequently one ion transmits its charge to another ion or that two ions of opposite charge exchange their charge when they meet or after they have "combined". In any case such processes must take place at the boundary between two substances when a current passes from one into the other. If for instance n positively charged copper atoms are deposited from a salt solution onto a copper plate and one considers also that in the copper plate all electricity is bound to ions, it is necessary to assume that the charges pass to n atoms in the plate or that $\frac{1}{2}n$ of the deposited particles have exchanged their charge with $\frac{1}{2}n$ negatively charged copper atoms already in the electrode.

If, therefore, the assumption of this transport or exchange of ion charges—admittedly still a highly unelucidated process—is an indispensible part of any theory which assumes the transport of electricity by ions, then a steady electric current never consists of convection alone, at least not when the centres of two touching or

† v. Helmholtz, *Wied. Ann.* **48**, 389 (1893).

‡ Although in a different way, Mr. Koláček (*Wied. Ann.* **32**, 244, 429 [1887]) has also given an explanation of the dispersion of the electric waves in molecules. Also to be cited is the theory of Mr. Goldhammer (*Wied. Ann.* **7**, 93 [1892]).

§ Giese, *Wied. Ann.* **37**, 576 (1889).

combined particles are at the distance l apart. The movement of electricity then takes place without convection over a distance of the order l, and only if this distance is very small in comparison to those over which convection takes place is one mainly concerned with the latter phenomenon.

Mr. Giese if of the opinion that a true convection does not take place at all in metals. As it seems, however, impossible to include the "jumping over" of charges in the theory, I beg to be excused if I myself completely avoid such a process and simply imagine a current in a copper wire as a movement of charged particles.

Further research will have to decide whether under a different assumption the results of the theory remain valid.

§ 3. The ionic theory was very appropriate for my purpose as it makes it possible to introduce the permeability for ether in a fairly satisfactory manner into the equations. These fall, of course, into two groups. First it is necessary to express how the state of the ether is determined by the charge, position and movement of the ions. Then, secondly, it must be noted with what forces the ether acts upon the charged particles. In my paper quoted above† I have derived the formulae from a few assumptions by means of D'Alembert's principle and thus chosen a way which is rather similar to Maxwell's use of Lagrange's equations. Now I prefer for brevity's sake to represent the fundamental equations themselves as hypotheses.

The formulae for the ether agree, as regards the space between the ions, with the well known equations of Maxwell's theory, and state in general that any change which is produced in the ether by an ion is propagated with the speed of light. However, we consider the force with which the ether acts on a charged particle to be dependent on the state of this medium at the position of the particle. The assumed fundamental law therefore differs in one essential point from the laws formulated by Weber and Clausius. The in-

† Lorentz, *La théorie électromagnétique de Maxwell et son application aux corps mouvants.*

fluence suffered by a particle B as a result of the proximity of a second particle A does depend on the movement of the latter, however not on its movement at that time. On the contrary the movement which this A had at a previous time is decisive, and the postulated law corresponds to the requirement which Gauss demanded in 1845 of the theory of electrodynamics which he expressed in his well known letter to Weber.†

Generally, we find, in the suppositions that I introduce, a return in a certain sense to the older theory of electricity. The kernel of Maxwell's views is not lost thereby, but it cannot be denied that one who subscribes to the ionic hypothesis is not very far removed from the notion of electrical particles which earlier theorists embraced. In certain simple cases this is particularly evident. Since we see the presence of electrical charge in an aggregate of positively or negatively charged particles, and our basic formulae yield Coulomb's Law for motionless ions, all, e.g., of electrostatics, can now be brought back to the earlier form.

† Gauss, *Werke*, **5**, 629.

11. SIMPLIFIED THEORY OF ELECTRICAL AND OPTICAL PHENOMENA IN MOVING SYSTEMS*

H. A. LORENTZ

§ 1. In former investigations I have assumed that, in all electrical and optical phenomena, taking place in ponderable matter, we have to do with small charged particles or ions, having determinate positions of equilibrium in dielectrics, but free to move in conductors except in so far as there is a resistance, depending on their velocities. According to these views an electric current in a conductor is to be considered as a progressive motion of the ions, and a dielectric polarization in a non-conductor as a displacement of the ions from their positions of equilibrium. The ions were supposed to be perfectly permeable to the aether, so that they can move while the aether remains at rest. I applied to the aether the ordinary electromagnetic equations, and to the ions certain other equations which seemed to present themselves rather naturally. In this way I arrived at a system of formulae which were found sufficient to account for a number of phenomena.

In the course of the investigation some artifices served to shorten the mathematical treatment. I shall now show that the theory may be still further simplified if the fundamental equations are immediately transformed in an appropriate manner.

§ 2. I shall start from the same hypotheses and introduce the

* *Proc. Roy. Acad. Amsterdam* **1**, 427 (1899).

same notations as in my "Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern". Thus, $\mathfrak{d}$ and $\mathfrak{H}$ will represent the dielectric displacement and the magnetic force, ϱ the density to which the ponderable matter is charged, $\mathfrak{v}$ the velocity of this matter, and $\mathfrak{E}$ the force acting on it per unit charge (electric force). It is only in the interior of the ions that the density ϱ differs from 0; for simplicity's sake I shall take it to be a continuous function of the coordinates, even at the surface of the ions. Finally, I suppose that each element of an ion retains its charge while it moves.

If, now, V be the velocity of light in the aether, the fundamental equations will be

$$Div\, \mathfrak{d} = \varrho, \tag{Ia}$$

$$Div\, \mathfrak{H} = 0, \tag{IIa}$$

$$Rot\, \mathfrak{H} = 4\pi\varrho\mathfrak{v} + 4\pi\dot{\mathfrak{d}}, \tag{IIIa}$$

$$4\pi V^2\, Rot\, \mathfrak{d} = -\dot{\mathfrak{H}}, \tag{IVa}$$

$$\mathfrak{E} = 4\pi V^2\mathfrak{d} + [\mathfrak{v}\,.\,\mathfrak{H}]. \tag{Va}$$

§ 3. We shall apply these equations to a system of bodies, having a common velocity of translation $\mathfrak{p}$, of constant direction and magnitude, the aether remaining at rest, and we shall henceforth denote by $\mathfrak{v}$, not the whole velocity of a material element, but the velocity it may have in addition to $\mathfrak{p}$

Now it is natural to use a system of axes of coordinates, which partakes of the translation $\mathfrak{p}$. If we give to the axis of x the direction of the translation, so that $\mathfrak{p}_y$ and $\mathfrak{p}_z$ are 0, the equations (Ia)-(Va) will have to be replaced by

$$Div\, \mathfrak{d} = \varrho, \tag{Ib}$$

$$Div\, \mathfrak{H} = 0, \tag{IIb}$$

$$\left.\begin{aligned}
\frac{\partial\mathfrak{H}_z}{\partial y} - \frac{\partial\mathfrak{H}_y}{\partial z} &= 4\pi\varrho(\mathfrak{p}_x+\mathfrak{v}_x) + 4\pi\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{d}_x,\\
\frac{\partial\mathfrak{H}_x}{\partial z} - \frac{\partial\mathfrak{H}_z}{\partial x} &= 4\pi\varrho\mathfrak{v}_y + 4\pi\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{d}_y,\\
\frac{\partial\mathfrak{H}_y}{\partial x} - \frac{\partial\mathfrak{H}_x}{\partial y} &= 4\pi\varrho\mathfrak{v}_z + 4\pi\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{d}_z,
\end{aligned}\right\} \tag{IIIb}$$

$$\left.\begin{aligned}
4\pi V^2\left(\frac{\partial\mathfrak{d}_z}{\partial y} - \frac{\partial\mathfrak{d}_y}{\partial z}\right) &= -\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{H}_x,\\
4\pi V^2\left(\frac{\partial\mathfrak{d}_x}{\partial z} - \frac{\partial\mathfrak{d}_z}{\partial x}\right) &= -\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{H}_y,\\
4\pi V^2\left(\frac{\partial\mathfrak{d}_y}{\partial x} - \frac{\partial\mathfrak{d}_x}{\partial y}\right) &= -\left(\frac{\partial}{\partial t} - \mathfrak{p}_x\frac{\partial}{\partial x}\right)\mathfrak{H}_z,
\end{aligned}\right\} \tag{IVb}$$

$$\mathfrak{E} = 4\pi V^2\mathfrak{d} + [\mathfrak{p}\cdot\mathfrak{H}] + [\mathfrak{v}\cdot\mathfrak{H}]. \tag{Vb}$$

In these formulae the sign *Div*, applied to a vector $\mathfrak{A}$, has still the meaning defined by

$$Div\, \mathfrak{A} = \frac{\partial\mathfrak{A}_x}{\partial x} + \frac{\partial\mathfrak{A}_y}{\partial y} + \frac{\partial\mathfrak{A}_z}{\partial z}.$$

As has already been said, $\mathfrak{v}$ is the relative velocity with regard to the moving axes of coordinates. If $\mathfrak{v} = 0$, we shall speak of a system at rest; this expression therefore means relative rest with regard to the moving axes.

In most applications $\mathfrak{p}$ would be the velocity of the earth in its yearly motion.

§ 4. Now, in order to simplify the equations, the following quantities may be taken as independent variables

$$x' = \frac{V}{\sqrt{(V^2-\mathfrak{p}_x^2)}}\,x, \quad y' = y, \quad z' = z, \quad t' = t - \frac{\mathfrak{p}_x}{V^2-\mathfrak{p}_x^2}\,x. \tag{1}$$

The last of these is the time, reckoned from an instant that is not the same for all points of space, but depends on the place we wish to consider. We may call it the *local time*, to distinguish it from the *universal time t*.

If we put

$$\frac{V}{\sqrt{(V^2-\mathfrak{p}_x^2)}} = k,$$

we shall have

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial x'} - k^2\frac{\mathfrak{p}_x}{V^2}\frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'},$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'}.$$

The expression

$$\frac{\partial \mathfrak{A}_x}{\partial x'} + \frac{\partial \mathfrak{A}_y}{\partial y'} + \frac{\partial \mathfrak{A}_z}{\partial z'}$$

will be denoted by

$$Div'\,\mathfrak{A}.$$

We shall also introduce, as new dependent variables instead of the components of $\mathfrak{d}$ and $\mathfrak{H}$, those of two other vectors $\mathfrak{F}'$ and $\mathfrak{H}$, which we define as follows

$$\mathfrak{F}'_x = 4\pi V^2\mathfrak{d}_x, \quad \mathfrak{F}'_y = 4\pi k V^2\mathfrak{d}_y - k\mathfrak{p}_x\mathfrak{H}_z, \quad \mathfrak{F}'_z = 4\pi k V^2\mathfrak{d}_z + k\mathfrak{p}_x\mathfrak{H}_y,$$
$$\mathfrak{H}'_x = k\mathfrak{H}_x, \quad \mathfrak{H}'_y = k^2\mathfrak{H}_y + 4\pi k^2\mathfrak{p}_x\mathfrak{d}_z, \quad \mathfrak{H}'_z = k^2\mathfrak{H}_z - 4\pi k^2\mathfrak{p}_x\mathfrak{d}_y.$$

In this way I find by transformation and mutual combination of the equations (Ib)–(Vb):

$$Div'\,\mathfrak{F}' = \frac{4\pi}{k}V^2\varrho - 4\pi k\mathfrak{p}_x\varrho\mathfrak{v}_x, \tag{Ic}$$

$$Div'\,\mathfrak{H}' = 0, \tag{IIc}$$

$$\left.\begin{aligned}
\frac{\partial \mathfrak{H}'_z}{\partial y'} - \frac{\partial \mathfrak{H}'_y}{\partial z'} &= 4\pi k^2\varrho\mathfrak{v}_x + \frac{k^2}{V^2}\frac{\partial \mathfrak{F}'_x}{\partial t'}\\
\frac{\partial \mathfrak{H}'_x}{\partial z'} - \frac{\partial \mathfrak{H}'_z}{\partial x'} &= 4\pi k\varrho\mathfrak{v}_y + \frac{k^2}{V^2}\frac{\partial \mathfrak{F}'_y}{\partial t'}\\
\frac{\partial \mathfrak{H}'_y}{\partial x'} - \frac{\partial \mathfrak{H}'_x}{\partial y'} &= 4\pi k\varrho\mathfrak{v}_z + \frac{k^2}{V^2}\frac{\partial \mathfrak{F}'_z}{\partial t'}
\end{aligned}\right\}, \tag{IIIc}$$

$$\left.\begin{aligned}
\frac{\partial \mathfrak{F}'_z}{\partial y'} - \frac{\partial \mathfrak{F}'_y}{\partial z'} &= -\frac{\partial \mathfrak{H}'_x}{\partial t'}\\
\frac{\partial \mathfrak{F}'_x}{\partial z'} - \frac{\partial \mathfrak{F}'_z}{\partial x'} &= -\frac{\partial \mathfrak{H}'_y}{\partial t'}\\
\frac{\partial \mathfrak{F}'_y}{\partial x'} - \frac{\partial \mathfrak{F}'_x}{\partial y'} &= -\frac{\partial \mathfrak{H}'_z}{\partial t'}
\end{aligned}\right\}, \tag{IVc}$$

$$\left.\begin{aligned}
\mathfrak{E}_x &= \mathfrak{F}'_x + k\frac{\mathfrak{p}_x}{V^2}(\mathfrak{v}_y\mathfrak{F}'_y + \mathfrak{v}_z\mathfrak{F}'_z) + (\mathfrak{v}_y\mathfrak{H}'_z - \mathfrak{v}_z\mathfrak{H}'_y)\\
\mathfrak{E}_y &= \frac{1}{k}\mathfrak{F}'_y - k\frac{\mathfrak{p}_x}{V^2}\mathfrak{v}_x\mathfrak{F}'_y + \left(\frac{1}{k}\mathfrak{v}_z\mathfrak{H}'_x - \mathfrak{v}_x\mathfrak{H}'_z\right)\\
\mathfrak{E}_z &= \frac{1}{k}\mathfrak{F}'_z - k\frac{\mathfrak{p}_x}{V^2}\mathfrak{v}_x\mathfrak{F}'_z + \left(\mathfrak{v}_x\mathfrak{H}'_y - \frac{1}{k}\mathfrak{v}_y\mathfrak{H}'_x\right)
\end{aligned}\right\}. \tag{Vc}$$

Putting $\mathfrak{v} = 0$ in the three last equations we see that

$$\mathfrak{F}'_x, \quad \frac{1}{k}\mathfrak{F}'_y, \quad \frac{1}{k}\mathfrak{F}'_z$$

are the components of the electric force that would act on a particle at rest.

§ 5. We shall begin with an application of the equations to electrostatic phenomena. In these we have $\mathfrak{v} = 0$ and $\mathfrak{F}'$ independent of the time. Hence, by (IIc) and (IIIc)

$$\mathfrak{H}' = 0,$$

and by (IVc) and (Ic)

$$\frac{\partial \mathfrak{F}'_z}{\partial y'} - \frac{\partial \mathfrak{F}'_y}{\partial z'} = 0, \quad \frac{\partial \mathfrak{F}'_x}{\partial z'} - \frac{\partial \mathfrak{F}'_z}{\partial x'} = 0, \quad \frac{\partial \mathfrak{F}'_y}{\partial x'} - \frac{\partial \mathfrak{F}'_x}{\partial y'} = 0,$$

$$Div'\,\mathfrak{F}' = \frac{4\pi}{k}V^2\varrho.$$

These equations show that $\mathfrak{F}'$ depends on a potential ω, so that

$$\mathfrak{F}'_x = -\frac{\partial \omega}{\partial x'}, \quad \mathfrak{F}'_y = -\frac{\partial \omega}{\partial y'}, \quad \mathfrak{F}'_z = -\frac{\partial \omega}{\partial z'}$$

and

$$\frac{\partial^2\omega}{\partial x'^2}+\frac{\partial^2\omega}{\partial y'^2}+\frac{\partial^2\omega}{\partial z'^2}=-\frac{4\pi}{k}V^2\varrho. \tag{2}$$

Let S be the system of ions with the translation $\mathfrak{p}_x$, to which the above formulae are applied. We can conceive a second system S_0 with no translation and consequently no motion at all; we shall suppose that S is changed into S_0 by a dilatation in which the dimensions parallel to OX are changed in ratio of 1 to k, the dimensions perpendicular to OX remaining what they were. Moreover we shall attribute equal charges to corresponding volume-elements in S and S_0; if then ϱ_0 be the density in a point P of S, the density in the corresponding point P_0 of S_0 will be

$$\varrho_0=\frac{1}{k}\varrho.$$

If x, y, z are the coordinates of P, the quantities x', y', z', determined by (1), may be considered as the coordinates of P_0.

In the system S_0, the electric force, which we shall call $\mathfrak{E}_0$ may evidently be derived from a potential ω_0, by means of the equations

$$\mathfrak{E}_{0x}=-\frac{\partial\omega_0}{\partial x'},\quad \mathfrak{E}_{0y}=-\frac{\partial\omega_0}{\partial y'},\quad \mathfrak{E}_{0z}=-\frac{\partial\omega_0}{\partial z'},$$

and the function ω_0 itself will satisfy the condition

$$\frac{\partial^2\omega_0}{\partial x'^2}+\frac{\partial^2\omega_0}{\partial y'^2}+\frac{\partial^2\omega_0}{\partial z'^2}=-4\pi V^2\varrho_0=-\frac{4\pi}{k}V^2\varrho.$$

Comparing this with (2), we see that in corresponding points

$$\omega=\omega_0,$$

and consequently

$$\mathfrak{F}'_x=\mathfrak{E}_{0x},\quad \mathfrak{F}'_y=\mathfrak{E}_{0y},\quad \mathfrak{F}'_z=\mathfrak{E}_{0z}.$$

In virtue of what has been remarked at the end of § 4, the components of the electric force in the system S will therefore be

$$\mathfrak{E}_{0x},\quad \frac{1}{k}\mathfrak{E}_{0y},\quad \frac{1}{k}\mathfrak{E}_{0z}.$$

Parallel to OX we have the same electric force in S and S_0, but in a direction perpendicular to OX the electric force in S will be $1/k$ times the electric force in S_0.

By means of this result every electrostatic problem for a moving system may be reduced to a similar problem for a system at rest; only the dimensions in the direction of translation must be slightly different in the two systems. If, e.g., we wish to determine in what way innumerable ions will distribute themselves over a moving conductor C, we have to solve the same problem for a conductor C_0, having no translation. It is easy to show that if the dimensions of C_0 and C differ from each other in the way that has been indicated, the electric force in one case will be perpendicular to the surface of C, as soon as, in the other case, the force $\mathfrak{E}_0$ is normal to the surface of C_0.

Since

$$k=\left(1-\frac{\mathfrak{p}_x^2}{V^2}\right)^{-\frac{1}{2}}$$

exceeds unity only by a quantity of the second order—if we call $\mathfrak{p}_x/V$ of the first order—the influence of the Earth's yearly motion on electrostatic phenomena will likewise be of the second order.

§ 6. We shall now shew how our general equations (Ic)—(Vc) may be applied to optical phenomena. For this purpose we consider a system of ponderable bodies, the ions in which are capable of vibrating about determinate positions of equilibrium. If the system be traversed by waves of light, there will be oscillations of the ions, accompanied by electric vibrations in the aether. For convenience of treatment we shall suppose that, in the absence of lighl-waves, there is no motion at all; this amounts to ignoring all molecular motion.

Our first step will be to omit all terms of the second order. Thus, we shall put $k = 1$, and the electric force acting on ions at rest will become $\mathfrak{F}'$ itself.

We shall further introduce certain restrictions, by means of which we get rid of the last term in (Ic) and of the terms containing $\mathfrak{v}_x, \mathfrak{v}_y, \mathfrak{v}_z$ in (Vc).

The first of these restrictions relates to the magnitude of the displacements $\mathfrak{a}$ from the positions of equilibrium. We shall suppose them to be exceedingly small, even relatively to the dimensions of the ions and we shall on this ground neglect all quantities which are of the second order with respect to $\mathfrak{a}$.

It is easily seen that, in consequence of the displacements, the electric density in a fixed point will no longer have its original value ϱ_0, but will have become

$$\varrho = \varrho_0 - \frac{\partial}{\partial x}(\varrho_0\mathfrak{a}_x) - \frac{\partial}{\partial y}(\varrho_0\mathfrak{a}_y) - \frac{\partial}{\partial z}(\varrho_0\mathfrak{a}_z).$$

Here, the last terms, which evidently must be taken into account, have the order of magnitude $c\varrho_0/a$, if c denotes the amplitude of the vibrations; consequently, the first term of the right-hand member of (Ic) will contain quantities of the order

$$\frac{V^2 c\varrho_0}{a}. \tag{3}$$

On the other hand, if T is the time of vibration, the last term in (Ic) will be of the order

$$\frac{\mathfrak{p}_x \varrho_0 c}{T}. \tag{4}$$

Dividing this by (3), we get

$$\frac{\mathfrak{p}_x}{V} \cdot \frac{a}{VT},$$

an extremely small quantity, because the diameter of the ions is a very small fraction of the wave-length. This is the reason why we may omit the last term in (Ic).

As to the equations (Vc), it must be remarked that, if the displacements are infinitely small, the same will be true of the velocities and, in general, of all quantities which do not exist as long as the system is at rest and are entirely produced by the motion. Such are $\mathfrak{H}'_x, \mathfrak{H}'_y, \mathfrak{H}'_z$. We may therefore omit the last terms in (Vc), as being of the second order.

The same reasoning would apply to the terms containing $\mathfrak{p}_x/V^2$, if we could be sure that in the state of equilibrium there are no electric forces at all. If, however, in the absence of any vibrations, the vector $\mathfrak{F}'$ has already a certain value $\mathfrak{F}'_0$, it will only be the difference $\mathfrak{F}' - \mathfrak{F}'_0$, that may be called infinitely small; it will then be permitted to replace $\mathfrak{F}'_y$ and $\mathfrak{F}'_z$ by $\mathfrak{F}'_{0y}$ and $\mathfrak{F}'_{0z}$.

Another restriction consists in supposing that an ion is incapable of any motion but a translation as a whole, and that, in the position of equilibrium, though its parts may be acted on by electric forces, as has just been said, yet the whole ion does not experience a resultant electric force. Then, if $d\tau$ is an element of volume, and the integrations are extended all over the ion,

$$\int \varrho_0 \mathfrak{F}'_{0y}\, d\tau = \int \varrho_0 \mathfrak{F}'_{0z}\, d\tau = 0. \tag{5}$$

Again, in the case of vibrations, the equations (Vc) will only serve to calculate the resultant force acting on an ion. In the direction of the axis of y e.g. this force will be

$$\int \varrho \mathfrak{F}_y\, d\tau.$$

Its value may be found, if we begin by applying the second of the three equations to each point of the ion, always for the same universal time t, and then integrate. From the second term on the right-hand side we find

$$-\frac{\mathfrak{p}_x}{V^2}\int \varrho\mathfrak{v}_x\mathfrak{F}'_y\,d\tau,$$

or, since we may replace $\mathfrak{F}'_y$ by $\mathfrak{F}'_{0y}$ and ϱ by ϱ_0,

$$-\frac{\mathfrak{p}_x}{V^2}\mathfrak{v}_x\int \varrho_0\mathfrak{F}'_{0y}\,d\tau,$$

which vanishes on account of (5).

Hence, as far as regards the resultant force, we may put $\mathfrak{E} = \mathfrak{F}'$, that is to say, we may take $\mathfrak{F}'$ as the electric force, acting not only on ions at rest, but also on moving ions.

The equations will be somewhat simplified, if, instead of $\mathfrak{F}'$, we introduce the already mentioned difference $\mathfrak{F}'-\mathfrak{F}'_0$. In order to do this, we have only twice to write down the equations (Ic)–(IVc), once for the vibrating system and a second time for the same system in a state of rest; and then to subtract the equations of the second system from those of the first. In the resulting equations, I shall, for the sake of brevity, write $\mathfrak{F}'$ instead of $\mathfrak{F}'-\mathfrak{F}'_0$, so that henceforth $\mathfrak{F}'$ will denote not the total electric force, but only the part of it that is due to the vibrations. At the same time we shall replace the value of ϱ, given above, by

$$\varrho_0-\mathfrak{a}_x\frac{\partial\varrho_0}{\partial x'}-\mathfrak{a}_y\frac{\partial\varrho_0}{\partial y'}-\mathfrak{a}_z\frac{\partial\varrho_0}{\partial z'}.$$

We may do so, because we have supposed $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$ to have the same values all over an ion, and because ϱ_0 is independent of the time, so that

$$\frac{\partial\varrho_0}{\partial x}=\frac{\partial\varrho_0}{\partial x'}.$$

Finally we have

$$Div'\,\mathfrak{F}' = -4\pi V^2\left(\mathfrak{a}_x\frac{\partial\varrho_0}{\partial x'}+\mathfrak{a}_y\frac{\partial\varrho_0}{\partial y'}+\mathfrak{a}_z\frac{\partial\varrho_0}{\partial z'}\right), \qquad \text{(Id)}$$

$$Div'\,\mathfrak{H}' = 0, \qquad \text{(IId)}$$

$$\frac{\partial\mathfrak{H}'_z}{\partial y'}-\frac{\partial\mathfrak{H}'_y}{\partial z'} = 4\pi\varrho_0\frac{\partial\mathfrak{a}_x}{\partial t'}+\frac{1}{V^2}\frac{\partial\mathfrak{F}'_x}{\partial t'}, \quad \text{etc.} \qquad \text{(IIId)}$$

$$\frac{\partial\mathfrak{F}'_z}{\partial y'}-\frac{\partial\mathfrak{F}'_y}{\partial z'} = -\frac{\partial\mathfrak{H}'_x}{\partial t'}, \quad \text{etc.} \qquad \text{(IVd)}$$

Since these equations do no longer explicitly contain the velocity $\mathfrak{p}_x$, they will hold, without any change of form, for a system that has no translation, in which case, of course, t' would be the same thing as the universal time t.

Yet, strictly speaking, there would be a slight difference in the formulae, when applied to the two cases. In the system without a translation $\mathfrak{a}_x$, $\mathfrak{a}_y$, $\mathfrak{a}_z$ would be, in all points of an ion, the same functions of t', i.e. of the universal time, whereas, in the moving system, these components would not depend in the same way on t' in different parts of the ion, just because they must everywhere be the same functions of t.

However, we may ignore this difference, of the ions are so small, that we may assign to each of them a single local time, applicable to all its parts.

The equality of form of the electromagnetic equations for the two cases of which we have spoken will serve to simplify to a large extent our investigation. However, it should be kept in mind, that, to the equations (Id)–(IVd) we must add the equations of motion for the ions themselves. In establishing these, we have to take into account, not only the electric forces, but also all other forces acting on the ions. We shall call these latter the molecular forces and we shall begin by supposing them to be sensible only at such small distances, that two particles of matter, acting on each other, may be said to have the same local time.

§ 7. Let us now imagine two systems of ponderable bodies, the one S with a translation, and the other one S_0 without such a motion, but equal to each other in all other respects. Since we neglect quantities of the order $\mathfrak{p}_x^2/V^2$, the electric force will, by § 5 be the same in both systems, as long as there are no vibrations.

After these have been excited, we shall have for both systems the equations (Id)–(IVd).

Further we shall imagine motions of such a kind, that, if in a point (x', y', z') of S_0 we find a certain quantity of matter or a certain electric charge at the universal time t', an equal quantity of matter or an equal charge will be found in the corresponding point of S at the local time t'. Of course, this involves that at these corresponding times we shall have, in the point (x', y', z') of both systems, the same electric density, the same displacement $\mathfrak{a}$, and equal velocities and accelerations.

Thus, some of the dependent variables in our equations (Id)–(IVd) will be represented in S_0 and S by the same functions of x', y', z', t', whence we conclude that the equations will be satisfied by values of $\mathfrak{H}'_x, \mathfrak{H}'_y, \mathfrak{H}'_z, \mathfrak{F}'_x, \mathfrak{F}'_y, \mathfrak{F}'_z$, which are likewise in both cases the same functions of x', y', z', t'. By what has been said at the beginning of this §, not only $\mathfrak{F}'$, but also the total electric force will be the same in S_0 and S, always provided that corresponding ions at corresponding times (i.e. for equal values of t') be considered.

As to the molecular forces, acting on an ion, they are confined to a certain small space surrounding it, and by what has been said in § 6, the difference of local times within this space may be neglected. Moreover, if equal spaces of this kind are considered in S_0 and S, there will be, at corresponding times, in both the same distribution of matter. This is a consequence of what has been supposed concerning the two motions.

Now, the simplest assumption we can make on the molecular forces is this, that they are *not* changed by the translation of the system. If this be admitted, it appears from the above considerations that corresponding ions in S_0 and S will be acted on by the same molecular forces, as well as by the same electric forces. Therefore, since the masses and accelerations are the same, the supposed motion in S will be possible as soon as the corresponding

motion in S_0 can really exist. In this way we are led to the following theorem.

If, in a body or a system of bodies, without a translation, a system of vibrations be given, in which the displacements of the ions and the components of $\mathfrak{F}'$ and $\mathfrak{H}'$ are certain functions of the coordinates and the time, then, if a translation be given to the system, there can exist vibrations, in which the displacements and the components of $\mathfrak{F}'$ and $\mathfrak{H}'$ are the same functions of the coordinates and the *local* time. This is the theorem, to which I have been led in a much more troublesome way in my "Versuch einer Theorie, etc.", and by which most of the phenomena, belonging to the theory of aberration may be explained.

§ 8. In what precedes, the molecular forces have been supposed to be confined to excessively small distances. If two particles of matter were to act upon each other at such a distance that the difference of their local times might not be neglected, the theorem would no longer be true in the case of molecular forces that are not altered at all by the translation. However, one soon perceives that the theorem would again hold good, if these forces were changed by the translation in a definite way, in such a way namely that the action between two quantities of matter were determined, not by the *simultaneous* values of their coordinates, but by their values at *equal local times*. If therefore, we should meet with phenomena, in which the difference of the local times for mutually acting particles might have a sensible influence, and in which yet observation showed the above theorem to be true, this would indicate a modification, like the one we have just specified, of the molecular forces by the influence of a translation. Of course, such a modification would only be possible, if the molecular forces were no direct actions at a distance, but were propagated by the aether in a similar way as the electromagnetic actions. Perhaps the rotation of the plane of polarization in the so-called active bodies will be found to be a phenomenon of the kind just mentioned.

§ 9. Hitherto all quantities of the order $\mathfrak{p}_x^2/V^2$ have been neglected.

As is well known, these must be taken into account in the discussion of MICHELSON's experiment, in which two rays of light interfered after having traversed rather long paths, the one parallel to the direction of the earth's motion, and the other perpendicular to it. In order to explain the negative result of this experiment FITZGERALD and myself have supposed that, in consequence of the translation, the dimensions of the solid bodies serving to support the optical apparatus, are altered in a certain ratio.

Some time ago, M. LIÉNARD† has emitted the opinion that, according to my theory, the experiment should have a positive result, if it were modified in so far that the rays had to pass through a solid or a liquid dielectric.

It is impossible to say with certainty what would be observed in such a case, for, if the explication of MICHELSON's result which I have proposed is accepted, we must also assume that the mutual distances of the molecules of transparent media are altered by the translation.

Besides, we must keep in view the possibility of an influence, be it of the second order, of the translation on the molecular forces.

In what follows I shall shew, not that the result of the experiment must necessarily be negative, but that this might very well be the case. At the same time it will appear what would be the theoretical meaning of such a result.

Let us return again to the equations (Ic)–(Vc). This time we shall not put in them $k = 1$, but the other simplifications of which we have spoken in § 6 will again be introduced. We shall now have to distinguish between the vectors $\mathfrak{E}$ and $\mathfrak{F}'$, the former alone being the electric force. By both signs I shall now denote, not the whole vector, but the part that is due to the vibrations.

The equations may again be written in a form in which the velocity of translation does not explicitly appear. For this purpose, it is necessary to replace the variables x', y', z', t', $\mathfrak{F}'$, $\mathfrak{H}'$, a and ϱ_0

† L'Éclairage Électrique, 20 et 27 août 1898.

by new ones, differing from the original quantities by certain constant factors.

For the sake of uniformity of notation all these new variables will be distinguished by double accents. Let ε be an indeterminate coefficient, differing from unity by a quantity of the order $\mathfrak{p}_x^2/V^2$, and let us put

$$x = \frac{\varepsilon}{k}x'', \quad y = \varepsilon y'', \quad z = \varepsilon z'', \tag{6}$$

$$\mathfrak{a}_x = \frac{\varepsilon}{k}\mathfrak{a}_x'', \quad \mathfrak{a}_y = \varepsilon\,\mathfrak{a}_y'', \quad \mathfrak{a}_z = \varepsilon\,\mathfrak{a}_z'', \tag{7}$$

$$\varrho_0 = \frac{k}{\varepsilon^3}\varrho_0'', \tag{8}$$

$$\mathfrak{F}_x' = \frac{1}{\varepsilon^2}\mathfrak{F}_x'', \quad \mathfrak{F}_y' = \frac{1}{\varepsilon^2}\mathfrak{F}_y'', \quad \mathfrak{F}_z' = \frac{1}{\varepsilon^2}\mathfrak{F}_z'',$$

$$\mathfrak{H}_x = \frac{k}{\varepsilon^2}\mathfrak{H}_x'', \quad \mathfrak{H}_y' = \frac{k}{\varepsilon^2}\mathfrak{H}_y'', \quad \mathfrak{H}_z' = \frac{k}{\varepsilon^2}\mathfrak{H}_z'',$$

$$t' = k\varepsilon t'', \tag{9}$$

so that t'' is a modified local time; then we find

$$Div''\,\mathfrak{F}'' = 4\pi V^2\left(-\mathfrak{a}_x''\frac{\partial\varrho_0''}{\partial x''} - \mathfrak{a}_y''\frac{\partial\varrho_0''}{\partial y''} - \mathfrak{a}_z''\frac{\partial\varrho_0''}{\partial z''}\right), \tag{Ie}$$

$$Div''\,\mathfrak{H}'' = 0, \tag{IIe}$$

$$\frac{\partial\mathfrak{H}_z''}{\partial y''} - \frac{\partial\mathfrak{H}_y''}{\partial z''} = 4\pi\varrho_0''\frac{\partial\mathfrak{a}_x''}{\partial t''} + \frac{1}{V^2}\frac{\partial\mathfrak{F}_x''}{\partial t''}, \quad \text{etc.} \tag{IIIe}$$

$$\frac{\partial\mathfrak{F}_z''}{\partial y''} - \frac{\partial\mathfrak{F}_y''}{\partial z''} = -\frac{\partial\mathfrak{H}_x''}{\partial t''}, \quad \text{etc.} \tag{IVe}$$

$$\mathfrak{E}_x = \frac{1}{\varepsilon^2}\mathfrak{F}_x'', \quad \mathfrak{E}_y = \frac{1}{k\varepsilon^2}\mathfrak{F}_y'', \quad \mathfrak{E}_z = \frac{1}{k\varepsilon^2}\mathfrak{F}_z''. \tag{Ve}$$

These formulae will also hold for a system without translation; only, in this case we must take $k = 1$, and we shall likewise take

$\varepsilon = 1$, though this is not necessary. Thus, x'', y'', z'' will then be the coordinates, t'' the same thing as t, i.e. the universal time, $\mathfrak{a}''$ the displacement, ϱ_0'' the electric density, $\mathfrak{H}''$ and ψ'' the magnetic and electric forces, the last in so far as it is due to the vibrations.

Our next object will be to ascertain under what conditions, now that we retain the terms with $\mathfrak{p}_x^2/V^2$, two systems S and S_0, the first having a translation, and the second having none, may be in vibratory states that are related to each other in some definite way. This investigation resembles much the one that has been given in § 7; it may therefore be expressed in somewhat shorter terms.

To begin with, we shall agree upon the degree of similarity there shall be between the two systems in their states of equilibrium. In this respect we define S by saying that the system S_0 may be changed into it by means of the dilatations indicated by (6); we shall suppose that, in undergoing these dilatations, each element of volume retains its ponderable matter, as well as its charge. It is easily seen that this agrees with the relation (8).

We shall not only suppose that the system S_0 *may* be changed in this way into an imaginary system S, but that, as soon as the translation is given to it, the transformation *really* takes place, of itself, i.e. by the action of the forces acting between the particles of the system, and the aether. Thus, after all, S will be the *same* material system as S.

The transformation of which I have now spoken, is precisely such a one as is required in my explication of MICHELSON's experiment. In this explication the factor ε may be left indeterminate. We need hardly remark that for the real transformation produced by a translatory motion, the factor should have a definite value. I see, however, no means to determine it.

Before we proceed further, a word on the electric forces in S and S_0 in their states of equilibrium. If $\varepsilon = 1$, the relation between these forces will be given by the equations of § 5. Now ε indicates an alteration of all dimensions in the same ratio, and it is very easy to see what influence this will have on the electric forces.

Thus, it will be found that, in passing from S_0 to S, the electric force in the direction of OX will be changed in the ratio of 1 to $1/\varepsilon^2$, and that the corresponding ratio for the other components will be as 1 to $1/k\varepsilon^2$.

As to the corresponding vibratory motions, we shall require that at corresponding times, i.e. for equal values of t'', the configuration of S may always be got from that of S_0 by the above mentioned dilatations. Then, it appears from (7) that $\mathfrak{a}_x''$, $\mathfrak{a}_y''$, $\mathfrak{a}_z''$ will be, in both systems, the same functions of x'', y'', z'', t'', whence we conclude that the equations (Ie)–(IVe) can be satisfied by values of $\mathfrak{F}_x''$, $\mathfrak{H}_x''$, etc., which are likewise, in S_0 and in S, the same functions of x'', y'', z'', t''.

Always provided that we start from a vibratory motion in S_0 that can really exist, we have now arrived at a motion in S, that is possible in so far as it satisfies the electromagnetic equations. The last stage of our reasoning will be to attend to the molecular forces. In S_0 we imagine again, around one of the ions, the same small space we have considered in § 7, and to which the molecular forces acting on the ion are confined; in the other system we shall now conceive the corresponding small space, i.e. the space that may be derived from the first one by applying to it the dilatations (6). As before, we shall suppose these spaces to be so small that in the second of them there is no necessity to distinguish the local times in its different parts; then we may say that in the two spaces there will be, at corresponding times, corresponding distributions of matter.

We have already seen that, in the states of equilibrium, the electric forces parallel to OX, OY, OZ, existing in S differ from the corresponding forces in S_0 by the factors

$$\frac{1}{\varepsilon^2}, \quad \frac{1}{k\varepsilon^2} \quad \text{and} \quad \frac{1}{k\varepsilon^2}.$$

From (Ve) it appears that the same factors come into play when we consider the part of the electric forces that is due to the vibrations. If, now, we suppose that the molecular forces are modified in quite the same way in consequence of the translation, we may apply the just mentioned factors to the components of the *total* force acting on an ion. Then, the imagined motion in S will be a possible one, provided that these same factors to which we have been led in examining the forces present themselves again, when we treat of the product of the masses and the accelerations.

According to our suppositions, the accelerations in the directions of OX, OY, OZ in S are resp. $1/k^3\varepsilon$, $1/k^2\varepsilon$ and $1/k^2\varepsilon$ times what they are in S_0. If therefore the required agreement is to exist with regard to the vibrations parallel to OX, the ratio of the masses of the ions in S and S_0 should be k^3/ε; on the contrary we find for this ratio k/ε, if we consider in the same way the forces and the accelerations in the directions of OY and OZ.

Since k is different from unity, these values cannot both be 1; consequently, states of motion, related to each other in the way we have indicated, will only be possible if in the transformation of S_0 into S the masses of the ions change; even this must take place in such a way that the same ion will have different masses for vibrations parallel and perpendicular to the velocity of translation.

Such a hypothesis seems very startling at first sight. Nevertheless we need not wholly reject it. Indeed, as is well known, the *effective* mass of an ion depends on what goes on in the aether; it may therefore very well be altered by a translation and even to different degrees for vibrations of different directions.

If the hypothesis might be taken for granted, MICHELSON's experiment should always give a negative result, whatever transparent media were placed on the path of the rays of light, and even if one of these went through air, and the other, say through glass. This is seen by remarking that the correspondence between the two motions we have examined is such that, if in S_0 we had a certain distribution of light and dark (interference-bands) we should have in S a similar distribution, which might be got from that in S_0 by the dilatations (6), provided however that in S the time of vibration

be $k\varepsilon$ times as great as in S_0. The necessity of this last difference follows from (9). Now the number $k\varepsilon$ would be the same in all positions we can give to the apparatus; therefore, if we continue to use the *same* sort of light, while rotating the instruments, the interference-bands will never leave the parts of the ponderable system, e.g. the lines of a micrometer, with which they coincided at first.

We shall conclude by remarking that the alteration of the molecular forces that has been spoken of in this § would be one of the second order, so that we have not come into contradiction with what has been said in § 7.

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